A TREATISE
ON
MILLS.

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A TREATISE ON MILLS,

IN FOUR PARTS.

PART FIRST,
ON CIRCULAR MOTION.

PART SECOND,
ON THE MAXIMUM OF MOVING BODIES, MACHINES, ENGINES, &C.

PART THIRD,
ON THE VELOCITY OF EFFLUENT WATER.

PART FOURTH,
EXPERIMENTS ON CIRCULAR MOTION, WATER-WHEELS, &C.

BY JOHN BANKS,
LECTURER IN EXPERIMENTAL PHILOSOPHY.

They continue this day according to thine ordinances.
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THE PREFACE.

THE experiments contained in the fourth part of this treatiſe, and ſome of theſe in the third, have been the ſubject of a public lecture, which I have occaſionally delivered, for about twenty years. —At the requeſt of many of my hearers, I have made it public; and have prefixed a few problems on circular motion, &c. For in order to know the powers of different wheels, or which is beſt able to overcome an obſtacle, it is neceſſary to know their central forces.

Where different powers are compared, as in Prob. XII, &c., they are ſuppoſed to act upon the wheel, while they deſcend through the ſame ſpace. When a wheel or fly, of any given weight or magnitude, has received a given degree of velocity, no more power is required to continue that velocity, than what is ſufficient to overcome the friction. Yet if it moves twice as faſt, it will require four times the former power to continue that motion: hence, one would be apt to infer, that the friction increaſes with the ſquare of the velocity.—In a water-wheel, the ſame power is not conſtant; for the ſame particles act upon the wheel during only a part of a revolution; their places are conſtantly ſupplied, or they are ſucceeded by others which act for their time, &c.: hence, a water-wheel ſoon acquires an uniform velocity.

In the ſecond part, the third problem has three ſolutions; the two firſt on principles the moſt common, but not perfectly juſt; but as they are ſufficiently near in practice, and much eaſier than the third, they are continued till the eighth problem, incluſive, which, in fact, is but one problem, for the fourth, &c. are illuſtrations of the theorems drawn from the third, and ought rather to have been examples. In this, as well as in many of the following problems, I have endeavoured to ſhew the impoſſibility of a given power producing more than a given effect, or of raiſing more than a certain quantity of water, weight, &c. to a given altitude, in a limited time. For, in this caſe, combinations of machinery afford no aſſiſtance, but the contrary. For if one pound in one ſcale cannot raiſe one pound in the other, we ſhall get nothing by ſuſpending a weight from the circumference of a wheel, in order to raiſe a greater at the circumference of its axle. For if the machine is ſo conſtructed that one pound will raiſe ten pounds, the pound muſt deſcend more than ten times as faſt as the other riſes, and of conſequence it muſt deſcend more than ten ſucceſſive times through a given ſpace, to raiſe ten pounds through the ſame ſpace.

It is true, that we have in the kingdom many intelligent engineers, and excellent mechanics; and there are others who can execute better than they can deſign, otherwiſe there would not have been ſo much money expended in attempting what men of ſcience know to be impoſſible.

When a man tells me he can conſtruct a water-wheel in ſuch a manner that, when once put in motion, it ſhall raiſe water to keep itſelf moving; or that he has conſtructed a pump in ſuch a manner, that one man may do the work of ten, &c. I pay the ſame attention to him, as if he told me he could create a ſyſtem of worlds, and command them to move. Or, is he leſs to be credited who ſays he can communicate perpetual motion to dead matter? Both are indirectly ſaying, I can reverſe the laws of nature.

Many of the problems in this part are intended not only to prove that a given power can only produce a given effect, but at the ſame time to demonſtrate what the greateſt effect is, which a given power can produce. For a power is often applied in ſuch a manner as to cauſe the effect to fall ſhort of the maximum, as demonſtrated in the third problem, when the time is given, and in the thirty-fifth, &c., when the ſpace is given, where it appears, that if the power and reſiſtance be equally diſtant from the center, in caſe of a maximum, or when the power produces the greateſt poſſible effect, the former will be to the latter as 1000 to 618. Or if the power be applied ten times as far from the center as the reſiſtance, when reduced they will be as 1000 to 659, the weight of the machine not conſidered, for which ſee Prob. XXXVII, XXXVIII, &c. When the power and reſiſtance are both given, by Prob. XXX, a machine may be conſtructed, by which the power ſhall produce the greateſt effect in a given time.

In the third part, the experiments on effluent water differ conſiderably from any theory; and, contrary to expectation, more water is diſcharged, in proportion, from ſmall than from large apertures. —The quantity through ſmall holes, I have endeavoured to aſcertain, by different proceſſes, all of which bring out nearly the ſame concluſions. And though I have had much practice in making experiments, I have not truſted entirely to my own obſervations, but have been aſſiſted in the whole, by one or more gentlemen well acquainted with the ſubject. At Coventry, by the Rev. Mr. Banks, of Monks Kirkby, by Meſſrs. Baines, Watſon, Decas, &c., of Liverpool, by Mr. Prieſtley, and Mr. Peckover, of Bradford, by my eldeſt ſon, and by my wife, who, *though a woman*, is perhaps as accurate in making experiments in philoſophy, and ſome branches of chymiſtry, as moſt of men.—Upon the whole, I preſume the rules contained in this part, for finding the quantity of water diſcharged in a given time, will be found ſufficiently accurate in practice.

However ſatisfactory mathematical reaſoning may be to ſome, yet experimental proof is deſirable, and, to many, much more ſo than the former; and without experiments, we often want *data* to reaſon from. But if we have certain principles, the concluſions drawn therefrom will often differ conſiderably from experiment, or rather the experiment from the theory. For the theory ſuppoſes the bodies to move in free ſpace, without friction, or any kind of reſiſtance; but as theſe impediments cannot be entirely removed, the experiments cannot perfectly coincide with the theory, though in ſome caſes they come exceedingly near.

In the fourth part, the experiments on circular motion, &c. are ſufficiently accurate to prove the truth of the theory, and the utility thereof, when applied to the conſtruction of machines. I have the preceding theory and experiments a neceſſary introduction to the principal ſubject, the experiments on wheels, &c.; in which I have endeavoured to inveſtigate the truth, without a view to ſupport any particular ſyſtem, ſentiments, or opinion. And if ſome of the experiments happen to recommend any conſtruction or application different to the practice of ſome profeſſional men, it might be well to enquire into the foundation of their practice, whether it is ſupported by experiments, or whether it reſts upon the opinions of their predeceſſors. To reſt ſatisfied with the opinions of others, however great their reputation, tends much to retard the progreſs of knowledge; for error is often found in high places.

It will not be expected that I ſhould attempt to inſtruct the mechanic how to form his cogs, divide the wheels, proportion their diameters, bevels, &c., as numbers of men may be found who can both plan and execute theſe parts well. The chief end in view is, how to make the moſt of a given ſtream; and what the experiments recommend, has long ſince been put in practice, with acknowledged advantage.—If this treatiſe, in proportion to its ſale, proves equally uſeful with the public lecture, I ſhall be ſatisfied.

To conclude, if, through miſtake, I have advanced any thing erroneous in theory, or drawn any falſe concluſions from the experiments, I hope thoſe who diſcover them will candidly correct them.

Jan. 29, 1795.

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A TREATISE ON MILLS.

PART I.

OF THE LAWS OF CIRCULAR MOTION, THE RATIOS OF PROJECTILE AND CENTRIFUGAL FORCES, THE PERIODIC TIMES, &C.

FIG. I.

IN the larger circle

D = AD the diameter in feet.

V = AC the projectile force; or orbit velocity in feet.

A = BC = AE the ſpace through which the body would move towards the center, while it is deſcribing AC, or the central force compared with the projectile.

$F = BC$ the centrifugal force compared with gravity, or with the weight of the revolving body, when on the ſurface of the earth.

$T =$ the periodic time, or time of a revolution, in ſeconds.

$p =$ the time deſcribing AE or AC.

$S = 16$ feet, the ſpace through which a body falls in 1 ſecond.

$q = 3.1416$, the circumference of a circle, when the diameter is 1.

In the leſſer circle

$d = ad$ the diameter.
$v = ac$ the velocity.
$a = ae$ the central force, &c.
$f = ae = bc$.
$t =$ the time of a revolution.

Every body in motion endeavours to move in a ſtraight line; the force which cauſes it to leave that line is called the *centripetal*, and the reſiſtance which it affords, the *centrifugal* force.

1. If a body at A, moving towards B, is drawn to C; BC repreſents the centrifugal *force*, and AE the centripetal *force*, which are equal, and as AE is to AC, ſo is the centrifugal to the projectile force, or circular velocity.

2. From the property of the circle,

$$\text{As } AD : AC :: AC : AE = \frac{\overline{AC}^2}{AD} = A.$$

$$\text{Or, As } D : V :: V : \frac{V^2}{D} = A.$$

3. And in equable motion it will be,

As the time of a revolution is to the circumference of the orbit, ſo is any other time to the ſpace paſſed over in that time; or ſo is 1 ſecond to the velocity per ſecond.

Viz. As $T : Dq :: p : \frac{Dqp}{T} = V$.

And by ſubſtituting $\frac{Dqp}{T}$ for its equal AC or V we have, As $D : \frac{Dqp}{T} :: \frac{Dqp}{T} : \frac{Dq^2p^2}{T^2} = AE = A$.

Or, As $D : V :: V : \frac{V^2}{D} = AE = A$.

4. Alſo, As $1''$: $s :: p^2 : sp^2$, the ſpace through which a body near the ſurface of the earth would fall in the time p.

5. Then, to compare the centrifugal force with the gravity, or weight of the revolving body, which put $= 1$.

It will be, As $sp^2 : AE :: 1 : \frac{AE}{sp^2} = F$.

Or, As $sp^2 : \frac{Dq^2p^2}{T^2} :: 1 : \frac{Dq^2}{sT^2} = F$.

Or, As $sp^2 : \frac{V^2}{D} :: 1 : \frac{V^2}{sDp^2} = F$.

When the central force is equal to the weight of the body, $\frac{Dq^2}{sT^2} = 1$, and $sp^2 = 16$, if $p = 1$ ſecond; alſo $A = s$.

From theſe proportions we obtain the following Theorems:

Theo. 1. $A = \frac{V^2}{D} = \frac{Vq}{T} = \frac{Dq^2}{T^2} = SF.$

Theo. 2. $V = \sqrt{AD} = \frac{Dq}{T} = \sqrt{FDS}.$

Theo. 3. $T = \frac{Vq}{A} = \frac{Dq}{V} = q\sqrt{\frac{D}{A}} = q\sqrt{\frac{D}{SF}} = \frac{Vq}{SF}.$

Theo. 4. $D = \frac{V^2}{A} = \frac{V}{SF} = \frac{T^2A}{q^2} = \frac{SFT^2}{q^2}.$

Theo. 5. $F = \frac{Dq^2}{ST^2} = \frac{A^2D}{SV^2} = \frac{V^2}{SD} = \frac{A}{S}.$

6. If two bodies, in different circles, revolve in the ſame time, the velocities will be directly as the diameters of the circles. ($T = t.$)

For, As $Dq : V\ (AC) :: dq : v\ (ac) = \frac{Vd}{D}$;

and, As $D : d :: V : v = \frac{Vd}{D}.$ Q. E. D.

And, As $AC : AE :: ac : ae.$

Or, As $V : A :: v : a = \frac{Av}{V}.$

Or, As $V : F :: v : \frac{vF}{V} = f.$

Hence, As $V : v :: F : f :: D : d.$

From which it appears, that, if the times are equal, the central force is directly as the diameter.

7. If $V = v$, then $Dq : dq :: T : \frac{Td}{D} = t$.

Therefore, As $D : d :: T : t$.

And, As $\frac{D}{T} : \frac{d}{t} : V : v = \frac{VdT}{Dt}$.

$\frac{D}{T} : \frac{d}{t} :: F : f = \frac{FdT}{Dt}$ and becauſe F is as $\frac{V^2}{D}$ we have, As $F : f :: \frac{V^2}{D} : \frac{v^2}{d}$. Or,

As $DF : df :: V^2 : v^2$, and $V : v :: \sqrt{DF} : \sqrt{df}$.

And $\frac{D}{T}$ being as V, we have $\frac{D}{T}\sqrt{df} = \frac{d}{t}\sqrt{DF}$.

Or, from the fifth Theorem, if $T = t$.

As $\frac{Dq^2}{ST^2} : \frac{dq^2}{ST^2} :: F : f = \frac{FD}{D}$.

Theo. 6. $v = \frac{Vd}{D}$.

Theo. 7. $d = \frac{vD}{V}$.

Theo. 8. $a = \frac{V^2 d}{D}$.

Theo. 9. $f = \frac{V^2 d}{SD}$.

8. If the diameter remains the ſame; or if $D = d$.

Then, As $\frac{Dq^2}{ST^2} : \frac{Dq^2}{ST^2} :: F : \frac{FT^2}{t^2} = f$, and as $t^2 : T^2 :: F : f :: V^2 : v^2$, or the central forces, in the ſame circle, are reciprocally as the ſquares of the times; or directly as the ſquares of the velocities.

Theo. 10. $f = \frac{FT^2}{t^2}$.

Theo. 11. $t = \sqrt{\frac{FT^2}{f}}$.

9. When $F = f$; $\frac{Dq^2}{ST^2} = \frac{dq^2}{st^2}$, or $Dt^2 = dT^2$.

And $\frac{D}{T^2} = \frac{d}{t^2}$ (ſee Theo. 5.) $\frac{Dt^2}{T^2} = d$.

And, As $T^2 : t^2 :: D : d$, or $T : t :: \sqrt{D} : \sqrt{d}$.

Theo. 12. $d = \frac{Dt^2}{T^2}$ when the times are given.

Theo. 13. $t = T\sqrt{\frac{d}{D}}$ when the diſtances are given.

The foregoing Theory exemplified, in the ſolution of various Problems in circular motion.

PROBLEM I.

Given the diameter of the orbit, 10 feet, and the centrifugal force equal to the weight of the revolving body; required the time of a revolution, and velocity per ſecond?

In this caſe, $A = S$; and $F = 1$ (ſee Art. 5.)

In Theo. 3. we have $\text{T} = q\sqrt{\frac{\text{D}}{s\text{F}}} = 3.1416\sqrt{\frac{10}{16}} = 2.4818''$; and per Theo. 2. $\text{V} = \frac{\text{D}q}{\text{T}} = \frac{31.416}{2.4818} = \sqrt{\text{FDS}} = \sqrt{160} = 12{\cdot}6491.$

PROBLEM II.

Given the time of a revolution, 3 ſeconds, the central force, 1; required the diſtance, and velocity?

$\left.\begin{array}{l}\text{T} = 3.\\ \text{F} = 1.\end{array}\right\}$ Theo. 4. $\text{D} = \frac{\text{T}^2 s}{q^2} = \frac{144}{9.8696} = 14.59$ feet; half of which is the diſtance, $= 7.295$ feet per ſecond.

And $\text{V} = \sqrt{\text{FDS}} = 15{\cdot}279.$

PROBLEM III.

Given the diameter and periodic time, to find the central force and velocity?

$\left.\begin{array}{l}\text{D} = 14.59.\\ \text{T} = 3''.\end{array}\right\}$ (Theo. 5.) $\text{F} = \frac{\text{D}q^2}{s\text{T}^2} = \frac{144}{144} = 1.$

$\text{V} = \frac{\text{D}q}{\text{T}} = \frac{45.837}{3} = 15.279$ feet.

PROBLEM IV.

Given the diameter, 14.59 feet, and the velocity, 15.279, to find the periodic time and central force?

(Per Theo. 3.) $\text{T} = \frac{\text{D}q}{\text{V}} = \frac{45.837}{15.279} = 3''$.

(Per Theo. 5.) $\text{F} = \frac{\text{V}^2}{\text{SD}} = \frac{233.44}{16 \times 14.59} = 1$.

PROBLEM V.

Given the diameter, 14.59 feet, the central force equal to twice the weight of the body; what is the velocity, and time of a revolution?

$\text{T} = q\sqrt{\frac{\text{D}}{2\text{FS}}} = \sqrt{\frac{14.59}{32}} \times 9 = .6753 \times 9 = 2{\cdot}1214''$.

$\text{V} = \sqrt{2\text{FDS}} = 21.6074$ feet per ſecond.

Or if we compare this problem with the laſt, we have (8)

As $f : \text{F} :: \text{T}^2 : t^2$, or $t = \text{T}\sqrt{\frac{\text{F}}{f}} = 3\sqrt{\frac{1}{2}} = 2.12$ &c.

And, As $\text{F} : f :: \text{V}^2 : v^2 = \frac{\text{V}^2 f}{\text{F}}$, or $v = \text{V}\sqrt{\frac{f}{\text{F}}} = 15.279 \times \sqrt{\frac{2}{1}} = 15.279 \times 1.414$ &c. $= 21.607$.

PROBLEM VI.

Let the diameter be 29.18 feet, (twice as much as in Prob. III.) the time of a revolution 3 ſeconds; required the velocity, and central force?

V =

$$V = \frac{Dq}{T} = \frac{91.671}{3} = 30.5572.$$

$$F = \frac{Dq^2}{ST^2} = \frac{288}{144} = 2.$$

(Art. 6.) As $D : d :: V : v = \frac{Vd}{D} = 30.5572.$

Viz. As $14.59 : 29.18 :: 15.279 : 30.5572.$

Again, As $D : d :: F : f = \frac{dF}{D}.$

As $14.59 : 29.18 :: 1 : 2$, the centrifugal force, the ſame as above.

PROBLEM VII.

The ſtones on which they grind table knives at Sheffield, are about 44 inches diameter, and weigh about half a ton; the velocity of the ſurface is at the rate of 1250 yards in a minute, equal to 326 revolutions; required the centrifugal force, or the tendency which the ſtones have to burſt?

$D = 2.44$ feet, the diameter of the circle of percuſſion.

$T = .184$ ſeconds, the time of one revolution.

$T^2 = .033856.$

$$F = \frac{Dq^2}{ST^2} = \frac{2.44 \times 9.8696}{16 \times .0338 \text{ \&c.}} = \frac{24.0818}{.54169} = 44.45$$

times the weight of the ſtone, or 22 tons.

PROBLEM VIII.

If a fly, 12 feet diameter, and 3 tons weight, revolves in 8 ſeconds, and another of the ſame weight revolves in 3 ſeconds; what muſt be the diameter of the laſt, when they have the ſame centrifugal force?

$D = 12.$
$T = 8.$
$t = 3.$
$F = f.$

And per Theo. 12. $d = \frac{Dt^2}{T^2} = \frac{108}{64} = 1.6875$ feet, the diameter of the circle of percuſſion.

N. B. As the weight of each fly is the ſame, it does not enter into the ſolution; but if the diameters ſhould be the ſame, as in the next Problem, the weight muſt be conſidered.

PROBLEM IX.

If a fly, 12 feet diameter, revolves in 8 ſeconds, and another of the ſame diameter in 3 ſeconds; what is the ratios of their weights, when the central forces are equal?

Here, $F = f$, and $D = d$, which put $= 1$.

And (Art. 8.) when $D = d$, we have, As $t^2 : T^2 :: F : f$; As $9 : 64 :: 1 : \frac{64}{9} = 7\frac{1}{9}$, the centrifugal force of the ſecond, when they are of equal weight.

And when the diſtance and time are the ſame, the force is directly as the weight; the weight of the ſecond will therefore be obtained by dividing the weight of the firſt by $7\frac{1}{9}$.

PROBLEM X.

If a fly, 2 tons weight, and 16 feet diameter, is ſufficient to regulate an engine, when it revolves in 4 ſeconds; what muſt be the weight of one 12 feet diameter, when it revolves in 2 ſeconds, ſo that it may have the ſame power upon the engine?

Here, we have, $D = 16$; $d = 12$; $T = 4$; $t = 2$; $F = f$; ſuppoſe d unknown, then, by Theo. 12, $d = \frac{Dt^2}{T^2} = 4$, when the weights are equal, viz. if the firſt is 16 feet diameter, and the ſecond 4 feet, their centrifugal forces would be equal; but, by the queſtion, the ſecond is to be 12 feet diameter, therefore, (whether it revolves in a circle of 4 feet or 12 feet, the time is 2 ſeconds) As $D : d :: F : f$, (Art. 6.) viz. As $4 : 12 :: 1 : 3 = f$, but $f = F$ by the queſtion, hence, the central force, and, of conſequence, the weight, is three times too great, and the ſecond fly ought to be one third of the weight of the firſt. $= \frac{40}{3} = 13\frac{1}{3}$ cwt.

Or, the two laſt Problems may be ſolved otherwiſe.

In Prob. IX. Let $w =$ the weight of the firſt fly $= 3$ tons. $x =$ the weight of the ſecond.

And

And (Art. 8.) we have, $t^2w = T^2x$, and $x = \frac{t^2w}{T^2}$ $= \frac{27}{64} = .4218 = 8.43$ cwt. $= \frac{60}{7\frac{1}{9}}$, as before.

In Prob. x. we have $\frac{Dt^2w}{dT^2} = x = \frac{128}{192} = .6666$ $= 13\frac{1}{3}$ cwt.

PROBLEM XI.

If a caſt-iron fly, 12 feet diameter, and 2 tons weight, has a ſufficient centrifugal force to regulate the pin ſhaft of a forge-hammer, when it revolves in 2 ſeconds; required the weight of a fly, of the ſame diameter, that ſhall have the ſame force, when it revolves in 4 ſeconds?

$T = 2$; $t = 4$; $D = d$; $F = f$; and $w = 2$; hence, $\frac{t^2w}{T^2} = x =$ the weight ſought $= \frac{16 \times 2}{4} =$ 8 tons, or four times the weight of the firſt.

N. B. When the weight of the revolving body and time of a revolution remain the ſame, the central force is directly as the diſtance from the center, or as the diameter of the wheel; (ſee Theo. 5.) but if the weight increaſes with the diſtance, or diameter, (and in water-wheels it often increaſes in a greater ratio) then, the centrifugal force will be as the ſquare of the diameter, viz. F is as D^2, $\frac{D^2}{T^2}$, or as $\frac{D^2q^2}{sT^2}$.

PROBLEM XII.

Let the centrifugal force of a wheel 6 feet diameter, and that of another of 12 feet diameter, be equal, or, in other words, let the impelling power be the ſame, and the firſt revolve in 5 ſeconds; required the time in which the ſecond revolves?

Here $F = f$; $D = 6$; $d = 12$; and $T = 5$.

By comparing Theo. 13. with the note to Prob. XI. we get $t = \frac{Td}{D} = 10$ ſeconds.

PROBLEM XIII.

Let the two wheels revolve in the ſame time; required the ratio of the impelling powers, or the central forces?

(See Art. 6.) As $D^2 : d^2 :: F : f = \frac{Fd^2}{D^2}$.

Let $F = 1$ and $\frac{Fd^2}{D^2} = \frac{144}{36} = 4$.

Viz. the 12 feet wheel requires four times as much power as the 6 feet wheel, to turn in the ſame time.

Otherwiſe, $F = \frac{D^2}{T^2}$; $f = \frac{d^2}{t^2}$; but, As $T = t$, $F = D^2$, and $f = d^2$; Or,

As $F : f :: 36 : 144 :: 1 : 4$ as before.

PROBLEM XIV.

Given the diameter of a water-wheel, 12 feet, its weight 2 tons, the time of a revolution 9 ſeconds; the diameter of the cog-wheel 10 feet, weight 12 cwt.; diameter of mill-ſtone 6 feet, weight one ton, time of a revolution ſix-tenths of a ſecond; required the centrifugal force of the water-wheel, cog-wheel, and mill-ſtone?

In the water-wheel, $D = 12$; $T = 9$; $w = 40$ cwt. And $\frac{Dq^2}{8T^2} \times w = F = \frac{4733}{1296} = 3.65$ cwt.

In the cog-wheel, $D = 10$; $T = 9$; $w = 12$. And $\frac{Dq^2w}{8T^2} = F = \frac{1183}{1296} = .91$.

In the mill-ſtone, $D = 4$ (the diameter of the circle of percuſſion;) $T = .6$; $w = 20$. And $\frac{Dq^2w}{8T^2} = F = \frac{789}{5.76} = 137$ cwt.

Water-wheel - - - - - - -	3.65
Cog-wheel - - - - - - - -	.91
Mill-ſtone - - - - - - - -	137.00

Centrifugal force of the whole - 141.56 = 7 tons 1.56 cwt.

In the above computation it is ſuppoſed, that the trundle, which turns the ſtone, is itſelf turned by the cog-wheel, its diameter therefore could not be more than 9 inches, for which reaſon, it is not taken into the computation.

PROBLEM XV.

Let the water-wheel and cog-wheel be the ſame as in the laſt Problem, the diameter of the trundle 2.5 feet, its weight 2 cwt. and on the ſame axis a cog or ſpur-wheel 10 feet diameter, and 12 cwt. which turns two trundles each 2.5 feet diameter, and weight together 4 cwt. the ſtones turned by them each 5 feet diameter, and weight together 36 cwt.; required the centrifugal force of the whole?

The central force of the water and cog-wheel, as in the laſt Problem, is = 4.56 cwt.

The diameter of the cog-wheel, being divided by that of the trundle, quotes 4, the number of revolutions that the trundle makes for 1 of the water-wheel, = 1 turn in $2\frac{1}{4}$ ſeconds.

And we may take $D = \frac{2}{3}$ of $2.5 = 1.66$; $T = 2.25$; $w = 2$. And $\frac{Dq^2w}{8T^2} = F = \frac{32.774}{81} = .404$ cwt. the force of the trundle.

In the ſpur-wheel, $D = 10$; $T = 2.25$; $w = 12$. And $\frac{Dq^2}{8T^2} \times w = \frac{1183.2}{81} = 14.6$ cwt. the force of the ſpur-wheel.

The ſecond trundle, or nut, will revolve four times for the ſpur-wheel once, therefore, if we divide 2.25, the time in which the latter revolves, by 4, we ſhall have the time in which the trundle and ſtone revolve, = .5625 ſecond.

$D = 1.66$; $T = .5625$; $w = 2$. And $\frac{Dq^2w}{8T^2} = F = 3.277$, which doubled, gives 6.554, the force of the two trundles.

Laſtly, for the force of the ſtones, we have $D = 3.33$; $T = .5625$; $w = 18$. And $\frac{Dq^2}{8T^2} = F = 117$, which, multiplied by 2, gives the centrifugal force of both mill-ſtones, = 234 cwt.

Force of the water and cog-wheel	4.56
Firſt trundle - - - - - - - -	.404
Spur-wheel - - - - - - - -	14.6
Second trundles - - - - - -	6.554
Mill-ſtones - - - - - - - -	234.
Central force of the whole -	250.118 cwt.

It may be obſerved, that the power which turns the wheel, or wheels, when applied in the ſame manner, has always the ſame ratio to the computed central force, and if a given power produces the forces computed in this Problem, the ratio of the power to turn it round, in any other time, is eaſily obtained, as follows:

PROBLEM XVI.

What power, compared with the laſt, would turn the wheel in 12 ſeconds?

Per Theo. 10. $t^2 : T^2 :: F : f = \frac{FT^2}{t^2}$.

Viz. As $12^2 : 9^2 :: 250 : 140$ nearly, or for every 250lb. in the firſt caſe, 140lb. will be ſufficient in the ſecond.

Obſervations on the quantity of water required to produce different velocities, in the ſame wheel, or train of wheels.

It has been demonſtrated (ſee Theo. 5. and 9. and Art. 8.) that in the ſame wheel, or wheels, the force is directly as the ſquare of the velocity: viz. if a weight, ſuſpended by a cord coiled round the axis of a wheel, can turn that wheel 10 times in a minute, it will require 4 times as much weight to turn it 20 times in one minute, or to communicate a double velocity. But if the power is a given ſtream of water, the caſe will be different.

Suppoſe a wheel is turned 10 times in a minute, and that every bucket receives a gallon of water (or any other quantity) if it is turned 20 times in a minute, a bucket can receive but half a gallon. Hence the ſtream muſt be doubled to ſupply every bucket with a gallon. But they ought to receive 4 gallons each. Therefore, the ſtream, when doubled, muſt be multiplied by 4, or made 8 times as large as at firſt, to ſupply the buckets with 4 times the quantity: conſequently the velocity of the wheel will be as the cube root of the quantity of water, when it acts by gravity.

PROBLEM XVII.

If a given quantity of water turns the drum, &c. of a cotton-mill 50 times in a minute; what quantity will be required to turn it 40 times in a minute?

Here the velocities are as the number of turns made in a given time, viz. as 50 to 40, or as 5 to 4; the cubes of which are as the quantities of water which produce them, viz. $5 \times 5 \times 5 = 125$, and $4 \times 4 \times 4 = 64$. Hence, for every 125 hogſheads in the firſt caſe, 64 will be ſufficient in the ſecond.

PROBLEM XVIII.

If a given quantity of water turns a wheel 6 times in a minute; how often will it be turned by double the quantity?

The velocities being as the cube root of the quantities, or as $\sqrt[3]{1} : \sqrt[3]{2}$, viz. as $1 : 1.2599$: and as one degree of velocity is to 6 turns, ſo is 1.2599 to 7.5594 per minute, the number of turns ſought.

A TREATISE

A TREATISE ON MILLS.

PART II.

TO INVESTIGATE THE VELOCITY OF A MACHINE; THE RATIO OF THE POWER AND RESISTANCE BEING GIVEN.

PROBLEM I.

FIG. II.

LET a weight a be ſuſpended at one end of a cord paſſing over a wheel, and a weight, or reſiſtance, x, at the other, to determine the ſpace paſſed over in a given time.

It is evident that $a + x$ is the quantity of matter to be moved, and that $a - x$ is the moving power.

EXAMPLE.

Let $a = 10$; $x = 5$; $s = 16$ feet, the ſpace fallen through by a heavy body in one ſecond; $v =$ the ſpace ſought.

Not taking the weight of the wheel into the expreſſion, we have,

As $a + x : a - x :: s : \frac{sa - sx}{a + x} = v =$ the ſpace paſſed over in the firſt ſecond of time, by the moving bodies, $= \frac{16 \times 10 - 16 \times 5}{10 + 5} = 5\frac{1}{3}$ feet.

PROBLEM II.

FIG. III.

Let W and X be two bodies connected by a cord; W hanging perpendicular, X ſupported by the inclined plane; required the accelerating force, and the ratio of X to W, when the effect is the greateſt.

Put $1 =$ radius.
$s =$ ſine of the angle of elevation.

Then as the radius (1) is to (s) the ſine of the angle of elevation, ſo is x to sx, the force with which the weight x endeavours to deſcend down the plane.

And as the ſum of their forces is to the difference, ſo is the force of gravity to the accelerating force.

Viz. As $w + sx : w - sx :: 1$ (the force of gravity) : $\frac{w - sx}{w + sx}$: the ſpace paſſed over in a given time, compared with gravity, or with the ſpace paſſed over by a falling body.

The greateſt effect of w will be when the momentum of x is a maximum, which will be had by multiplying the velocity by the quantity.

Viz. $\frac{w - sx}{w + sx} \times x =$ maximum.

Or, $\frac{wx - sx^2}{w + sx} =$ the momentum, and muſt be a maximum by the queſtion; in fluxions

$$\overline{w\dot{x} - 2sx\dot{x}} \times \overline{w + sx} - s\dot{x} \times \overline{wx - sx^2} = 0,$$

from which we get $s^2x^2 + 2swx = w^2$; by compleating the ſquare, and extracting the root, we have $x = \frac{w}{s}\sqrt{2} - \frac{w}{s}$.

EXAMPLE.

Let $w = 10$; $s = .5$ the natural ſine of 30°, to find x?

$\frac{w}{s}\sqrt{2} = 28.28$, from which ſubtract $\frac{w}{s}$ (20) and we have $8.28 = x$. In caſe of a pully, $s = 1$, and $x = w\sqrt{2} - w$.

PROBLEM III.

FIG. IV.

Given a weight w, ſuſpended from a wheel A, required the weight x, ſuſpended from the axis B, when the effect is a maximum?

Let BC $= a = 1$; AC $= b$; $s = 16$ feet; the velocity of $x = 1$; and as $a : b :: 1 : \frac{b}{a}$, and when $a = 1$, the velocity of $w = b$, and (per mechanics) the momentum of w will be $= wb$, and that of $x = a$, or x, when $a = 1$; therefore $wb + x =$ the ſum of the momentums, and as $wb + x : wb - x :: s : \frac{swb - sx}{wb + x} = v$, the velocity of w; and as AC : BC :: $b : a ::$ the velocity of w : the velocity of $x = \frac{swb - sx}{wb^2 + bx}$, which multiplied by x gives its momentum, $\frac{swbx - sx^2}{wb^2 + bx}$, which muſt be a maximum.

In fluxions, $\overline{bw\dot{x} - 2x\dot{x}} \times \overline{wb^2 + bx} - b\dot{x} \times \overline{wbx - x^2} = 0$, and $b^2w^2 - bwx - 2x^2 = bwx - x^2$; from which we find $x = bw \times \sqrt{2} -$

bw: viz. if bw is multiplied by the ſquare root of 2 (1.414214) and bw taken from the product, the remainder is x; or if 1 is taken from the ſquare root of 2, and the remainder (.414214) is multiplied by bw, the product is x.

Note, Whether we multiply the velocity of w, or the velocity of x, by x, to obtain the momentum, the value of x will be the ſame.

Otherwiſe, Let $v =$ the velocity of x, then As $a : v :: b : \frac{bv}{a} =$ the velocity of $w = bv$ when $a = 1$, and $\frac{x}{b} =$ the power which applied at B will balance x, hence $\frac{bw - x}{b}$ is the accelerating force $= f$; and $vx + \frac{x}{b} + f \times bv =$ the ſum of the momentums $= f = \frac{bw - x}{b}$, viz. $2xv + bfv = \frac{bw - x}{b}$; and by ſubſtituting $\frac{bw - x}{b}$ for f, we have $vx + bwv = \frac{bw - x}{b}$, and $v = \frac{bw - x}{b^2 w + bx}$; the ſame as before.

Again, the ſame notation remaining, $\frac{ax}{b} =$ the power which balances x, and $\frac{bw - x}{b} =$ the accelerating force at x, and $\frac{a^2 x}{b^2} + w =$ the whole reſiſtance reduced to B, by which divide $\frac{bw - x}{b}$ and we have $\frac{b^2w - bax}{b^2w + a^2x}$ for the force at B, which multiplied by x gives $\frac{b^2wx - bax^2}{b^2w + a^2x}$ for the momentum.

The fluxion of which is $\overline{b^2w\dot{x} - 2bax\dot{x}} \times \overline{b^2w + a^2x} - a^2\dot{x} \times \overline{b^2wx - bax^2} = 0$ and $x = \frac{bw}{a}\sqrt{\frac{b^2 + ba}{a^2}} - \frac{b^2w}{a^2}$; or if $a = 1$, $x = bw\sqrt{b^2 + b} - b^2w$.

Mathematicians in general have eſtimated the forces of moving bodies by multiplying the quantity into the velocity, as in the firſt and ſecond ſolutions of this Problem. The laſt ſolution is on different principles, except in finding the momentum, where the accelerating force is multiplied by the quantity; and though it gives the value of x exactly the ſame when a and b are equal, yet it varies a little when they are unequal. And the effect of w although a maximum, will vary with the ratio of $a : b$: for inſtance, if $w = 10$, $a = 1$, $b = 2$, the momentum of x will be 2.0204; but if $b = 10$, the effect will be 2.382, and if $b = 100$, x will be found 498, and its momentum 2.487.

If any three of the four terms a, b, w, x be given, the fourth may be found, when the effect is the greateſt, by the following Theorems.

Let $n = \sqrt{2} - 1 = .414214$.

Then, Theo. 1. $x = \frac{bwn}{a}$.

Theo. 2. $w = \frac{ax}{bn}$.

Theo. 3. $a = \frac{bwn}{x}$.

Theo. 4. $b = \frac{ax}{wn}$.

The two following Theorems give the velocities, whether the effect be a maximum or otherwiſe.

Theo. 5. $v = \frac{wb - xa}{wb + xa}$ = velocity of w.

Theo. 6. $v = \frac{wb - x}{wb + x} \times \frac{a}{b}$ = velocity of x.

If the velocities thus found be multiplied by 16, the product is the number of feet paſſed over in the firſt ſecond.

PROBLEM IV.

Given a, b, and w, to find x?

Let $a = 2$; $b = 3$; $w = 10$. Then,

(Theo. 1.) $x = \frac{bwn}{a} = \frac{3 \times 10 \times .4142}{2} = 6.21321$.

PROBLEM V.

Given a, b, and x, to find w?

Let $a = 10$; $b = 12$; and $x = 20$.

(Theo. 2.) $w = \frac{ax}{bn} = \frac{200}{4.9705} = 40.23$.

PROBLEM VI.

Given $a = 1$, $x = 100$, $w = 20$; required b, the diſtance of w?

(Theo. 4.) $b = \frac{ax}{wn} = \frac{100}{8.2842} = 12.071$.

PROBLEM VII.

Given $w = 100$, $b = 10$, $x = 500$; required a, the diſtance of x?

(Theo. 3.) $a = \frac{bwn}{x} = .82842$.

PROBLEM VIII.

The ſame terms given, to find the velocity of w?

(Theo. 5.) $v = \frac{wb - ax}{w + ax} = \frac{1000 - 414.21}{1000 + 414.21} = \frac{585.89}{1414.21} = .414 \times 16 = 6.6274$ feet the firſt ſecond.

In the following table, the power *w*, remains the ſame; the weight *x*, varies with the diſtance of *w*, and is found by the foregoing theorems; and the ſame power in every example produces the ſame momentum, viz. the weight *x*, multiplied by its velocity, gives 27.4516, which it can never exceed.

TABLE I.

No.	*a*	*x*	*b*	*w*	velocity of *x*.	momentum of *x*.
1	1	4.142	1	10	6.6274	27.4516 +
2	1	8.284	2	10	3.3137	27.4516
3	1	12.426	3	10	2.2091	27.4516
4	1	16.568	4	10	1.6568	27.4516
5	1	41.42	10	10	.6627	27.4516
6	1	82.84	20	10	.3313	27.4516

For if the diſtance of *w* is increaſed, the momentum of *x* will be diminiſhed.

For example, let *a*, *x*, and *w* be the ſame as in No. 1 of the table, and let $b = 2$; then $v = \frac{wb - x}{wb + x} = 10.5088$, which divided by *b*, gives 5.2544, the velocity of *x*, which multiplied by *x*, gives the momentum, $= 21.7637$, but in the table it is 27.45.

Let *a*, *b*, and *x* be varied as in Table II. while the power *w* remains the ſame; the true momentum, computed from the given term, is always leſs than the maximum, except No. 2.

TABLE II.

No.	*a*	*x*	*b*	*w*	Velocity of *x*.	Momentum of *x*.	Velocity of w.
1	1	10	2	10	2 666	26.66	5.333
2	1	10	2.4142	10	2.74516	27.4516	6.627
3	1	10	4	10	2.4	24	9.6
4	1	10	50	10	.30744	3.0744	15.372
5	1	10	1000	10	.01596	.1596	15.96
6	0	0	10	10	.0000	.000	16

The greateſt ſpace that w can poſſibly paſs over in the firſt ſecond is 16 feet; but if it has any reſiſtance to overcome it muſt fall ſhort of that ſpace, and that more or leſs as the circumſtances may be: and if the reſiſtance of the machine is not taken into conſideration, the effect is a maximum, or the greateſt, *when the momentum of the power is to the momentum of the weight as* 10 *to* 4.14214; *or as* 70 *to* 29, *nearly.*

In the above Problems, the accelerating force compared with gravity, always is as the difference of the weights to their ſum, or as the difference divided by the ſum.

But if the moving power, or force, acted according to any different law, the ratio of the power to the weight, in caſe of a maximum, would vary alſo. As for inſtance, where a wheel is impelled by the force of a given ſtream, and which force varies with the velocity of the wheel, viz. it is always as the ſquare of the velocity with which it ſtrikes the wheel, as demonſtrated in the next Problem.

PROBLEM IX.

Suppoſe the power to be the impulſe of a ſtream of water upon the floats of a wheel.

Let the quantity of water be repreſented by 1, and the velocity by 1, theſe multiplied together give the momentum; let the velocity be doubled, and it follows that the quantity will be doubled, for while the aperture remains the ſame, a double velocity muſt diſcharge a double quantity: hence, the momentum will always be as the ſquare of the velocity.

Let $v =$ velocity of the ſtream.

$x =$ velocity of the wheel.

$v - x =$ velocity with which the ſtream ſtrikes the wheel.

And $\overline{v - x}|^2$ will be the force with which the ſtream impels the wheel; this multiplied by the velocity of the wheel, gives the momentum thereof; and when the effect is greateſt it muſt be a maximum: hence, $\overline{v - x}|^2 \times x = M$, a maximum.

And $v^2\dot{x} - 4vx\dot{x} + 3x^2\dot{x} = 0$.

$x^2 - \frac{4vx}{3} = -\frac{v^2}{3}$; from which we get $x = \frac{v}{3}$.

And as $v^2 : \overline{v-x}|^2 :: 1 : \frac{\overline{v-x}|^2}{v^2} :: 9 : 4$.

EXAMPLE.

Let $v = 6$; then $x = 2$; and $v - x = 4$.

And as v^2 (36) :: $\overline{v-x}|^2$ (16) :: 1 : $\frac{4}{9}$.

Hence it appears, that when the velocity of a wheel, which is driven by impulſe only, is one third the velocity of the ſtream, the effect is the greateſt, and the power is to the reſiſtance as 9 to 4.

In the above proceſs the weight of the wheel is not conſidered; hence, the force acting upon it may be conſidered as the weight, which multiplied by the velocity, gives the relative effect, or momentum, and the concluſion is the ſame as if w had been taken to repreſent a weight, which, ſuſpended from the wheel, would balance the impulſe of the water.

In order to eſtimate the force of a ſtream upon an underſhot wheel, it has been common to take the area of the ſection, and depth, but this method is exceedingly erroneous (as will appear in the third part of this work) and perhaps we cannot, from any known principles, independent of experiments, compute the true force of a ſtriking fluid; we muſt therefore proceed with the relative force, which is as the ſquare of the *velocity*, while the aperture remains the ſame; but if the depth remains the ſame, and the apèrture varies, the force is directly as the area of the aperture.

In this Problem the effect produced is as $\overline{\mathrm{v}-x}|^2x$; let $\mathrm{v}=6$, $x=1$.

$$\left.\begin{array}{l}\text{Then } \overline{\mathrm{v}-x}|^2x = 25.\\ x=2.\ \text{Then } \overline{\mathrm{v}-x}|^2x = 32.\\ x=3.\ \text{Then } \overline{\mathrm{v}-x}|^2x = 27.\\ x=4.\ \text{Then } \overline{\mathrm{v}-x}|^2x = 16.\\ x=5.\ \text{Then } \overline{\mathrm{v}-x}|^2x = 5.\end{array}\right\}\ \text{The relative effects of the different velocities of the wheel.}$$

PROBLEM X.

FIG. V.

Given a lever equally thick, to find the length, PB, ſo that it may, by its own weight, juſt balance the end AP, and weight W?

Let z = AP.
x = PB, the length ſought.
m = the weight of 1 foot in length of the lever.

Then, zm = the weight of AP; and xm = the weight of PB: but by the queſtion, the momentum of PB muſt be equal to the momentum of AP + W.

The momentum of PB is $mx\dot{x}$, that is, the weight multiplied by the fluxion of the length; in like manner, the momentum of AP is $mz\dot{z}$, and the momentum of W is wz: hence, $mx\dot{x} = mz\dot{z} + wz$.

The fluent of which is $\frac{mx^2}{2} = \frac{mz^2}{2} + wz$; and $x^2 = z^2 + \frac{2wz}{m}$; and $x = \sqrt{z^2 + \frac{2wz}{m}}$.

Otherwiſe, without fluxions, let x = PB, the length ſought, as before, and make PC = AP $= z$; and PD = $(\frac{\text{CB}}{2} + \text{PC})\ \frac{x+z}{2}$ = the diſtance of the center of gravity of CB from the prop P, which multiplied by its weight, $mx - mz$, gives $\frac{mx^2 - mz^2}{2}$ for the momentum, which muſt be equal to zw; the momentum of AP is equal to that of PC, and therefore not conſidered: hence we have, $mx^2 - mz^2 = 2zw$; and $x = \sqrt{z^2 + \frac{2zw}{m}}$, as before.

Otherwiſe,

If the lever is ſuppoſed to move the mean velocity of the end, PB will be as $\frac{x}{2}$; that of AP as $\frac{z}{2}$; and that of w as z; and the quantities of matter are w, mz, and mx, which multiplied by their velocities gives $wz + \frac{mz^2}{2} = \frac{mx^2}{2}$, from which x is found $= \sqrt{\frac{2wz}{m} + z^2}$, as before.

From which we have the following Theorems.

Theo. 2. $z = \sqrt{x^2 + \frac{w^2}{m^2}} - \frac{w}{m}$.

Theo. 3. $w = \frac{mx^2}{2z} - \frac{mz}{2}$.

Theo. 4. $m = \frac{2wz}{x^2 - z^2}$.

EXAMPLE.

Let $z = 2$ feet, $w = 24$lbs. $m = 3$lbs.; required x, the length which will balance the ſhorter end, and weight w?

Theo. 1. $x = \sqrt{\frac{2wz}{m} + z^2} = \sqrt{\frac{96}{3} + 4} = 6$.

PROBLEM XI.

Given one end of a ſtraight bar or lever, equally thick, and homogeneous, 6 feet; the weight of a foot in length, 4 pounds; and a weight of 60 pounds ſuſpended from the ſhorter; required the length of the ſhorter, when it, together with the weight, will balance the longer?

Here, $x = 6$, $w = 60$, $m = 4$, to find z.

(Per Theo. 2.) $z = \sqrt{x^2 + \frac{w^2}{m^2}} - \frac{w}{m} = \sqrt{36 + \frac{3600}{16}} - \frac{60}{4} = 1.15$ feet, the diſtance of the weight from the prop.

PROBLEM XII.

Given the longer end 10 feet, the ſhorter 2 feet, the weight at the ſhorter end 100 pounds; required the weight of the lever, ſo that the whole may be in equilibrio?

$x = 10$ feet, $z = 2$ feet, $w = 100$lbs.; required m.

This is ſolved by Theo. 4, where $m = \frac{2wz}{x^2 - z^2} = \frac{400}{96} = 4.166$ pounds, the weight of 1 foot, which multiplied by 12, the whole length, gives 50 pounds for the whole weight of the lever.

PROBLEM XIII.

FIG. VI.

If a power P, at the diſtance AC, raiſes a weight W, at the diſtance DC, through DG, in a given time; what weight M muſt be placed at the diſtance BC, ſo that P may deſcend equally faſt, or that the angular velocity may be the ſame?

Let CB = CA = a; CD = b.

Agreeable to the laws of motion, a given power in a given time moves W through DG; but when a weight $\frac{Wb}{a}$ is applied at B, it is acted upon by a power (from the property of the lever) which is to that acting at D as $b : a$; and as $a : b :: W : \frac{bw}{a}$, hence the power and weight have ſtill the ſame ratio; of conſequence the power would urge the weight through the ſame ſpace in the ſame time; but the ſpace ought to be directly as the diſtance when P moves equally faſt, or when the angular velocity is the ſame; and the power being the ſame, the maſs muſt be increaſed in the ſame inverſe ratio as the velocity is diminiſhed, or as $b : a :: \frac{bw}{a} : \frac{b^2w}{a^2} =$ the weight which applied at B, would be raiſed through BE, by the power P, while W would be raiſed through DG by the ſame power; from which is derived the third ſolution to the third Problem.

Otherwiſe,

Let V = velocity of the point D.

Then, As $b : a :: V : \frac{Va}{b}$ = the velocity of the point B; but the power is able to communicate to $\frac{bw}{a}$, when placed at B, a velocity V. Therefore the effect produced is $\frac{Vbw}{a} = M \times \frac{Va}{b}$; or $MVa^2 = Vwb^2$, and $M = \frac{wb^2}{a^2}$, as before.

The ſame ſolution is obtained by a different proceſs, in Prob. XIII. part firſt, where D and d = the diſtances = a and b; and there As $D^2 : d^2 :: F$ to f: and if $F = W$, we have $a^2 : b^2$ as $W : \frac{b^2W}{a^2}$.

PROBLEM XIV.

FIG. VII.

Given the weight of a wheel ab, condenſed in the circumference; and a weight p ſuſpended by a cord coiled round the wheel; to determine the velocity of the weight, and of conſequence of the circumference of the wheel?

Put w = the weight of the wheel = 20.
$p = 1$. Then, it is evident, the maſs to be moved will be $w + p = 21$; and as the wheel is at liberty to reſt in any poſition, the moving power is only p.

Therefore, as $w + p : p$ ſo is the whole force of gravity, to that part which is exerted in moving the wheel: viz. As $w + p : p :: 1 : \frac{p}{w+p}$; or As $21 : 1 :: 1 : \frac{1}{21}$, which multiplied by 16, gives the feet paſſed over in the firſt ſecond $= \frac{16}{21} = .7619$ feet, or 9.14 inches.

EXAMPLE II.

Let $p = 2$; then $w + p = 22$, and the accelerating force will be $\frac{2}{22} = \frac{1}{11}$ or 1.4545 feet the firſt ſecond.

EXAMPLE III.

If $p = 3$; then $w + p = 23$, and $\frac{p}{w+p} = \frac{3}{23} \times 16 = 2.087$ feet.

EXAMPLE IV.

If $p = 4$; then $\frac{p}{w+p} = \frac{4}{24}$, which multiplied by 16, gives 2.66 feet.

In theſe examples, the maſs to be moved is increaſed by increaſing the power, as it makes part of the maſs: but in computing central forces, as in the firſt part of this work, the moving power or force is conſidered as diſtinct from the maſs moved, which, according to this

Problem, may be illuſtrated by ſuppoſing the maſs to diminiſh as the power increaſes.

EXAMPLE V.

Let the weight of the wheel be 31, and the power 1. Then $\frac{p}{w+p} = \frac{1}{32}$, which multiplied by 16, gives $\frac{16}{32} = \frac{1}{2}$ foot for the ſpace paſſed over in the firſt ſecond.

EXAMPLE VI.

Let the weight be 30; the power 2. Then $\frac{p}{w+p} = \frac{2}{32} = \frac{1}{16}$, or 1 foot in the firſt ſecond.

EXAMPLE VII.

Make the weight 29; the power 3. And we have $\frac{p}{w+p} = \frac{3}{32} = 1.5$ feet the firſt ſecond.

EXAMPLE VIII.

Let the weight be 28; the power 4. And $\frac{p}{w+p} = \frac{4}{32} = \frac{1}{8} = 2$ feet per ſecond.

EXAMPLE IX.

In the four laſt Examples, required the time in which the power will deſcend through 20 feet?

In falling bodies, as the ſpace is to the ſquare of the time, ſo is any other ſpace to the ſquare of the time in which it is deſcribed.

In Ex. 5. As ½ foot : $1^{2\prime\prime}$ :: 20 feet : $\overline{6.32}|^{2\prime\prime}$, which is the time required to fall through 20 feet.

In Ex. 6. As 1 foot : $1^{2\prime\prime}$ :: 20 feet : $\overline{4.47}|^{2\prime\prime}$.

In Ex. 7. As 1½ feet : $1^{2\prime\prime}$ :: 20 feet : $\overline{3.65}|^{2}$

In Ex. 8. As 2 feet : $1^{2\prime\prime}$ :: 20 feet : $\overline{3.16}|^{2}$.

In Examples 5th, 6th, 7th, and 8th the moving powers are as 1, 2, 3, and 4, and the ſpaces paſſed over in a given time are as 1, 2, 3, and 4; but the times in which a given ſpace is deſcribed are inverſely as the ſquare roots of the impelling powers, viz. in this caſe, as 1, 1.414, 1.73, and 2, inverſely, which numbers are the ſquare roots of the impelling powers. And it may be obſerved, that when the velocity of a revolving body is ſaid to be as the ſquare root of the impelling power, that power is ſuppoſed to deſcend through a given ſpace, which is always the caſe with water applied to turn a wheel.

PROBLEM XV.

FIG. VIII.

Given the power, and diſtance from the axis at which it is applied, and alſo the diameter and weight of the wheel, to determine the velocity of the power, and of the circumference of the wheel?

Let the diſtance at which the power acts be $= d$; the radius of the wheel $= r$; its weight $= w$.

Then, from the property of the lever, $\frac{dp}{r} =$ the force with which p acts upon the wheel, or it expreſſes the moving power. The maſs moved is $w + \frac{d^2p}{r^2}$ by which divide $\frac{dp}{r}$ and we have $\frac{dpr}{r^2w + d^2p}$ for the accelerating force at the circumference of the wheel.

And as $r : d :: \frac{dpr}{r^2w + d^2p} : \frac{d^2pr}{r^3w + d^2pr}$, which divided by r, gives $\frac{d^2p}{r^2w + d^2p}$ for the velocity of p.

EXAMPLE I.

Let $p = 4$; $d = 3$; $r = 9$; $w = 48$.

Then, $\frac{dpr}{r^2w + d^2p} = \frac{3 \times 4 \times 9}{81 \times 48 + 9 \times 4} = \frac{108}{3924} = \frac{3}{109}$ of gravity, which multiplied by 16, gives $\frac{48}{109}$ of a foot the firſt ſecond.

And as $9 : 3 :: \frac{48}{109} : \frac{16}{109}$, the ſpace paſſed over by p in the ſame time.

EXAMPLE

EXAMPLE II.

Again, let $r = 18$, the reſt as before. And we have $\frac{216}{15588} = \frac{6}{433}$ for the velocity of the wheel, which multiplied by 16, gives the feet paſſed over in the firſt ſecond.

In this Example, the power p, reduced to the wheel, does not afford ſo much reſiſtance as in the firſt; for there $\frac{d^2p}{r^2} = \frac{4}{9}$; but here $\frac{d^2p}{r^2} = \frac{1}{9}$; hence if w, in this Example, had been increaſed by $\frac{3}{9}$, the accelerating force would have been exactly half of what it is found in the firſt Example: but without that addition it differs $\frac{1}{8266}$, or is ſo much above one half of the other.

EXAMPLE III.

Let the diſtance $d = 1$; the radius of the wheel $r = 84$; the weight $w = 9$; $p = 8$; required the time in which p deſcends through 24 feet?

$\frac{dpr}{r^2w + d^2p} = \frac{84}{7939}$, which multiplied by 16, gives .16929 feet the firſt ſecond.

And as $r : d :: .16929 : .002015$, the feet paſſed over by p in the ſame time.

And as $.002015$ feet $: 1^{2''} ::$ 24 feet $: \overline{109.1}|^{2''}$, the time required.

On the Center of Gyration.

PROBLEM XVI.

FIG. IX.

Given a ſtraight bar AB, equally thick, &c. to find O, the center of gyration, which if ſtruck will communicate the ſame angular velocity to the bar as if the whole bar was collected in that point?

The force of any particle revolving round a center is as that particle multiplied by the ſquare of its velocity, or of its diſtance from the center of motion; of conſequence, the force required to deſtroy that motion muſt be equal to it.

Put AC $= z$; BC $= x$; CO $= y$, the center of gyration ſought.

Then $\dot{z}$ multiplied by the ſquare of its diſtance $z^2 = z^2\dot{z}$, and its fluent $\frac{z^3}{3} =$ the force of the end AC, and $\frac{x^3}{3}$ will be the force of the end BC; but when x and z are removed to O, their force will be $\overline{x+z} \times y^2$; which force muſt be equal to $\frac{x^3}{3} + \frac{z^3}{3}$: viz. $\overline{x+z} \times y^2 = \frac{x^3+z^3}{3}$; and $y = \sqrt{\frac{x^3+z^3}{3x+3z}}$.

EXAMPLE

EXAMPLE.

Let $z = 7$; $x = 14$. Then, $y^2 = \frac{2744 + 343}{42 + 21} = \frac{3087}{63}$, and $y = \sqrt{49} = 7$, the diſtance of the center of gyration from the center of motion.

PROBLEM XVII.

FIG. X.

Given three bodies A, B, and D, connected by a line ſuppoſed without weight, to find the diſtance of N the center of gyration, from C the center of motion?

Let $CA = a$; $CB = b$; $CD = d$; $CN = y$.

$Aa^2 + Bb^2 + Dd^2 = \overline{A + B + D} \times y^2$; for $\frac{Aa^2}{y^2} + \frac{Bb^2}{y^2} + \frac{Dd^2}{y^2} =$ the ſum of all the forces reduced to N, which, by the problem, muſt be equal to all the bodies placed at N and multiplied by $\overline{CN}^2$; hence $y = \sqrt{\frac{Aa^2 + Bb^2 + Dd^2}{A + B + D}}$.

EXAMPLE.

Let $A = 2$; $a = 10$; $B = 4$; $b = 6$; $D = 6$; $d = 4$.

Then, $\sqrt{\frac{200 + 144 + 96}{2 + 4 + 6}} = 6.05 = y$, the diſtance ſought.

PROBLEM XVIII.

FIG. XI.

Given the length and weight of a bar, two weights A and B placed at the ends, and C the center of motion, to find N the center of gyration?

Let $b =$ the weight of AC; $a =$ the weight of BC; $x =$ AC; $z =$ BC; $w =$ the weight at A; $p =$ the weight at B.

Then, $wx^2 + \frac{bx^2}{3} + pz^2 + \frac{az^2}{3} =$ the force of the whole revolving round C, to which $\overline{w + p + a + b} \times$ CN muſt be equal; hence, CN, or y, $= \sqrt{\frac{3wx^2 + bx^2 + 3pz^2 + az^2}{3 \times w + p + a + b}}$.

EXAMPLE.

Let BC $= a = 1$; AC $= b = 2$; the weight at A $= w = 1$; the weight at B $= p = 1$; $x = 14$; $z = 7$.

And we have $y = \sqrt{\frac{588 + 392 + 147 + 49}{3 \times 1 + 1 + 1 + 2}} = \sqrt{\frac{1176}{15}} = 8.85$, at which diſtance, if the whole maſs in the ſyſtem was collected, it would require the ſame force to give it the ſame angular motion, or to deſtroy the motion, as if the bodies had been fixed as in the Problem.

On

On the Center of Gravity.

THIS center, in any body or ſyſtem of bodies, if ſuſtained, the whole remains at reſt. This has been treated on by ſo many authors, and is in general ſo well underſtood, that a few obſervations will be ſufficient.

PROBLEM XIX.

FIG. XII.

Given two bodies A and B, connected by the line AB, to find C, their common center of gravity?

Let $AB = d$; $AC = x$.

Then, $A \times x = B \times \overline{a - x}$, or $Ax = Ba - Bx$; and $Ax + Bx = Ba$: hence, $x = \frac{Ba}{A + B}$.

EXAMPLE.

Let $AB = a = 100$; $A = 40$; $B = 10$.

Then, $\frac{100 \times 10}{40 + 10} = 20 = x$, the diſtance of the center of gravity from the larger body.

PROBLEM XX.

Given the length of an homogeneous beam, the weight of a foot in length, and a weight ſuſpended from one end, to find the center of gravity?

Let n = the length in feet.
m = the weight of a foot in length.
w = the weight ſuſpended.
x = the longer end.
z = the ſhorter end = $n - x$.

Then $mz\dot{z} + wz$ = the momentum of the ſhorter end; and $mx\dot{x}$ = the momentum of the longer end. And by making their fluents equal, according to the Problem, we have $\frac{mx^2}{2} = \frac{mz^2}{2} + wz$, and by ſubſtituting $n - x$ for z, we have $\frac{mx^2}{2} = \frac{mn^2 - 2mnx + mx^2}{2} + wn - wx$, from which x is found $= \frac{mn + 2w}{mn + w} \times \frac{n}{2}$.

EXAMPLE.

Let $n = 20$; $m = 100$; $w = 1000$.

Then we have $x = \frac{2000 + 2000}{2000 + 1000} \times \frac{20}{2} = 13\frac{3}{9}$, or $13\frac{1}{3}$.

PROBLEM XXI.

FIG. XIII.

Given the length of a beam AB $= 2n$; the weight of a foot in length $= m$; the weight W, and its diſtance from A $= a$; the weight P, and its diſtance from B $= b$, to find G, the center of gravity of the whole?

Let $x =$ AG; $z =$ BG $= 2n - x$.

Then the momentum of W will be $= w \times \overline{x - a}$, the momentum of the end x will be $= \frac{mx^2}{2}$, the momentum of P $= p \times \overline{z - b}$, and that of z, or its equal $2n - x$, $= \frac{4mn^2 - 4mnx + mx^2}{2}$; hence we have $mx^2 + 2wx - 2wa = 4mn^2 - 4mnx + mx^2 + 4pn - 2px - 2bp$; from which we find $x = \frac{2mn^2 + 2pn + wa - bp}{2mn + w + p}$.

If the weights W and P are at the ends of the beam, then will x and $2n - x =$ their diſtances from the center of gravity, and the theorem will be $x = \frac{4mn^2 + 4pn}{4mn + 2w + 2p} = \frac{mn + p}{2mn + w + p} \times 2n$.

EXAMPLE.

$n = 10$; $m = 4$; $w = 10$; $p = 3$; $a = 4$; $b = 1$. Then $x = \frac{1600 + 120 + 80 - 6}{160 + 20 + 6} = 9.6451$, the diſtance of the center of gravity from the end A.

Of

Of the Center of Percuſſion.

AS in bodies at reſt the whole weight may be conſidered as collected in the center of gravity, ſo in bodies in motion the whole force may be conſidered as concentrated in the center of percuſſion, or center of force.

PROBLEM XXII.

FIG. XIV.

In a ſtraight rod OA, required C, the center of percuſſion?

Put $a = \text{OA}$, the length of the rod; $x = \text{OC}$, the diſtance of the center of percuſſion from the point of ſuſpenſion; $\text{OG} =$ the diſtance of the center of gravity $= \frac{a}{2}$. And the weight of the rod multiplied by the diſtance of the center of gravity from the point O, will be equal to the force of the rod divided by the diſtance of the center of percuſſion from O.

Therefore, $\frac{a^2}{2} = \frac{a^3}{3x}$, and $3a^2x = 2a^3$, which divide by a^2, and we have $3x = 2a$, or $x = \frac{2a}{3}$.

And as the ſtroke is the ſame as if the whole weight of the rod was condenſed in this point, it is alſo the center of oſcillation; for, if it be conſidered as a pendulum, the whole force of the rod is exerted in the point C, to move the pendulum, and of conſequence the time of vibration muſt be the ſame as if the whole weight was actually collected in that point.

PROBLEM XXIII.

FIG. XV.

Let the center of motion, or point of ſuſpenſion, be at a diſtance from the end of the rod, to find the center of percuſſion?

Let OB $= a = 8$; OA $= b = 4$; OC $= x$.

Then, $\frac{a-b}{2} =$ OG, the diſtance of the center of gravity from O, which multiplied by the weight $a + b$ gives $\frac{a^2 - b^2}{2} = \frac{a^3 + b^3}{3x}$; and $3x \times \overline{a^2 - b^2} = 2 \times \overline{a^3 + b^3}$, or $x = \frac{a^3 + b^3}{a^2 - b^2} \times \frac{2}{3} = 8$; hence, in this Example, C coincides with B.

PROBLEM XXIV.

FIG. XVI.

Given the length and weight of a rod, and the weight of a ball at the end, to find the center of percuſſion, or oſcillation?

Put

Put $n =$ the length of the rod; $m =$ the weight of a foot; $w =$ the weight of the ball; $y =$ OG, the diſtance of the center of gravity from O; $x =$ OC, the diſtance of the center of oſcillation.

Then, $y = \frac{mn + 2w}{mn + w} \times \frac{n}{2}$; (ſee Prob. XX.) which multiplied by the weight $mn + w$ gives $\frac{mn^2 + 2wn}{2} = \frac{mn^3 + 3wn^2}{3x}$; and $x = \frac{mn^3 + 3wn^2}{mn^2 + 2wn} \times \frac{2}{3}$.

Otherwiſe,

The weight of the rod is mn, which multiplied by half its length, gives $\frac{mn^2}{2}$ for the moment of the rod; and $n \times w$ is the moment of the ball. Alſo, mn, the weight of the rod, multiplied by the ſquare of the length, n^2, gives mn^3, which divided by 3, gives the whole force of the rod. And $w \times n^2$ is the force of the ball, and the ſum of their forces divided by the ſum of their moments, gives the diſtance of the center of percuſſion from the point of ſuſpenſion, $= \frac{\frac{1}{3}mn^3 + wn^2}{\frac{1}{2}mn^2 + wn}$, the ſame as above.

PROBLEM XXV.

FIG. XVII.

Given the length and weight of a rod, and the weights and places of two balls fixed to it; required the center of percuſſion or oſcillation?

Put m = the weight of a foot; n = the length OW; a = OP; x = OC.

Then $\frac{mn^2}{2}$ = the momentum of the rod; that of the weight P is aP; and that of W = nW; the ſum of the whole is $\frac{mn^2}{2} + ap + aw$.

The force of the rod is $\frac{mn^3}{3}$; the force of the weight P is P × a^2; the force of the weight W is W × n^2; and the ſum is $\frac{mn^3}{3} + Pa^2 + Wn^2$, which divided by the ſum of the momentum, gives $x = \frac{mn^2 + 3Pa^2 + 3Wn^2}{mn^2 + 2Pa + 2wn} \times \frac{2}{3}$, the ſame as if $mn + W + P$, the whole weight of the rod and balls, were multiplied by $\frac{mn^2 + 2nW + 2aP}{2mn + 2W + 2P}$, the diſtance of the center of gravity from the point of ſuſpenſion.

Note, If the diſtance of the center of oſcillation from the center of the ſyſtem, or point of ſuſpenſion, be multiplied by the diſtance of the center of gravity from the ſame point; the ſquare root of the product will be the diſtance of the center of gyration: for inſtance,

$$\frac{mn^3+3\text{P}a^2+3\text{W}n^2}{mn^2+2\text{P}a+2\text{W}n} \times \frac{2}{3} \times \frac{mn^2+2\text{P}a+2n\text{W}}{2mn+2\text{P}+2\text{W}} =$$

$$\frac{mn^3+3\text{P}a^2+3\text{W}n^2}{2mn+2\text{P}+2\text{W}} \times \frac{2}{3} = \frac{mn^3+3\text{P}a^2+3\text{W}n^2}{3\times\overline{\text{W}+\text{P}+mn}},$$

which laſt expreſſion is the ſquare of the center of gyration, as found by Prob. XVIII.

EXAMPLE.

Let $n = 12 = \text{OW}$, $\text{W} = 2$; $a = 6 = \text{OP}$, $\text{P} = 2$; $m = 1$.

Then $\frac{mn^2+2\text{P}a+2\text{W}n}{2mn+2\text{P}+2\text{W}} = \frac{216}{32} = 6.75$, the diſtance of the center of gravity from the point O.

And $\frac{mn^3+3\text{P}a^2+3\text{W}n^2}{mn^2+2\text{P}a+2\text{W}n} \times \frac{2}{3} = \frac{5616}{648} = 8\frac{2}{3}$, the diſtance of the center of oſcillation or percuſſion.

Alſo, $\frac{mn^3+3\text{P}a^2+3\text{W}n^2}{3\times\overline{\text{W}+\text{P}+mn}} = \frac{2808}{48} = 58.5$, the ſquare root of which is 7.6485, the center of gyration.

Put $\text{G} = 6\frac{3}{4}$, the center of gravity.
$\text{P} = 8\frac{2}{3}$, the center of percuſſion.
$\text{C} = 7.6485$, the center of gyration.

Then $\text{GP} = \text{CC}$, and As $\text{G} : \text{C} :: \text{C} : \text{P} = \frac{\text{C}^2}{\text{G}} = \frac{58.5}{6.75} = 8\frac{2}{3}$; and $\text{P} : \text{C} :: \text{C} : \text{G} = \frac{\text{C}^2}{\text{P}} = \frac{58.5}{8\frac{2}{3}} = \frac{351}{52} = 6\frac{3}{4}$.

PROBLEM XXVI.

FIG. XVIII.

Given the length and weight of a rod, the center of ſuſpenſion S, a weight W at the bottom, and another P at the top, to find the center of percuſſion?

Let $m =$ the weight of a foot in length; $n =$ SW feet; $a =$ SP feet.

Then, the whole force of the rod and balls will be $\frac{mn^3 + 3\text{W}n^2 + ma^3 + 3\text{P}a^2}{3}$: the momentum of the end SW will be $\frac{mn^2}{2} + \text{W}n$: the momentum of the end PS will be $\frac{ma^2}{2} + a\text{P}$. And by dividing the ſum of their forces by the difference of their momentums we have $\frac{mn^3 + 3\text{W}n^2 + ma^3 + 3\text{P}a^2}{mn^2 + 2\text{W}n - ma^2 - 2\text{P}a}$ $\times \frac{2}{3} =$ the center of percuſſion ſought.

EXAMPLE.

Let $m = 16$; $\text{W} = \text{P} = 39$ dwts.; $a = \frac{1}{3}$ foot; $n = \frac{2}{3}$ foot. Then, from the above Theorem,

$$\frac{\frac{128}{27} + \frac{468}{9} + \frac{16}{27} + \frac{117}{9}}{\frac{64}{6} + \frac{156}{3} - \frac{16}{9} - \frac{78}{3}} \times \frac{2}{3} = \frac{1899}{864} \times \frac{2}{3} = 1.4964$$

feet, the diſtance of the center of percuſſion or oſcillation from the point of ſuſpenſion. C, the center of percuſſion, is, in this caſe, further removed from the point of ſuſpenſion than the weight W.

Note. If S is the center of the rod, or if SB = SW, and a = the diſtance of P, the expreſſion for the center of oſcillation will be a little ſhorter, viz. $= \frac{2mn^3 + 3Wn^2 + 3Pa^2}{2Wn - 2Pa} \times \frac{2}{3}$.

Let every thing be the ſame as in the laſt Example, only SB = SW = n. Then, from the laſt Theorem, the diſtance of the center of oſcillation from S is found = 22.917 inches.

Note 2. If the weight of the rod is not conſidered, the Theorem becomes $\frac{Wn^2 + Pa^2}{Wn - Pa}$ = SC, the diſtance of the center of oſcillation from S.

Note 3. If m = the weight of an inch, and the diſtances are given in inches, we have the anſwer in inches.

PROBLEM XXVII.

Given a ſtraight bar, equally heavy, to determine the point of ſuſpenſion, when the vibrations are performed in the leaſt time poſſible; or when the diſtance of the center of oſcillation from the axes of motion is a minimum.

Let $n =$ the whole length of the bar; $x =$ the length of the ſhorter end.

Then will $\frac{n^3 - 3n^2x + 3nx^2}{n^2 - 2nx} \times \frac{2}{3}$ expreſs the diſtance of the center of oſcillation from the point of ſuſpenſion, which is to be a minimum.

Its fluxion is $\overline{6nx\dot{x} - 3n^2\dot{x}} \times \overline{n^2 - 2nx} - 2n\dot{x} \times \overline{n^3 - 3n^2x + 3nx^2} = 0$. And $6nx - 3n^2 - 12x^2 + 6nx = 6nx - 2n^2 - 6x^2$; from which we get $6nx - 6x^2 = n^2$; and by compleating the ſquare we have $x^2 - nx + \frac{n^2}{4} = \frac{n^2}{4} - \frac{n^2}{6}$: and $x = \frac{n}{2} - n\sqrt{\frac{1}{12}}$.

EXAMPLE.

Let the length of the rod be 40 inches $= n$. Then $\frac{40}{2} - 40\sqrt{\frac{1}{12}} = 8.4532 = x$, the diſtance of the center of ſuſpenſion from the end, when it vibrates in the leaſt time poſſible.

The center of oſcillation will be found $=$ 23.0926 inches, equally diſtant from the lower end of the rod, as the center of ſuſpenſion is from the top thereof. Hence, if the center of oſcillation is made the center of motion, the center of motion becomes the center of oſcillation, and the time of vibration remains the ſame.

In a ſtraight rod, the center of oſcillation is two-thirds of its length from the top, when it is ſuſpended at the top; but if the bar is ſuſpended at one-third of its length from the end, the center of oſcillation is at the bottom, or lower end, and the vibrations are performed in the ſame time. The center found in this Problem would be the center of gyration, if the center of gravity were the center of ſuſpenſion. In this caſe, the center of gravity is in the middle of the rod; and the center of gyration is $n\sqrt{\frac{1}{12}}$, which taken from $\frac{n}{2}$ gives the center of ſuſpenſion, as before.

When a pendulum, or vibrating body, is placed in a horizontal poſition, the initial velocity of the center of oſcillation is the ſame as that produced by the action of gravity on bodies left to fall in ſtraight lines towards the earth.

Let AB (FIG. XIX.) be a pendulum, S the center of motion, B the center of oſcillation, then the end B will begin to deſcend equally faſt as a body unconnected with the pendulum. For, if the end AS affords reſiſtance, it throws the center of oſcillation to B, which if AS were removed would be at C. The point C would, in that caſe, begin to move with the velocity of a falling body, and B would move faſter as it is further from the center of motion. Or if the center of oſcillation were removed beyond the end of the vibrating body, (ſee Fig. XVIII.) the weight W would deſcend with a leſs velocity, or it would be As SC : SW :: 16 : 16 × $\frac{SW}{SC}$ = the initial velocity of W. Hence, in any ſyſtem, having found the centers of gravity and oſcillation, we find their tendency to motion, or the initial accelerating force.

PROBLEM XXVIII.

Given, the length of a beam, 20 feet; its weight, 200lb.; a weight, W, = 60lb. at one end; and a power P, = 100, at the other end; it is required to aſſign the velocity of P, which is equal to that of W when the beam, equally thick, is ſuſpended by the center of gravity?

A given power applied to any part of the beam will communicate the ſame velocity as if the whole maſs was collected in the center of gyration.

Let a = 10 feet, half the length of the beam.
m = 100lb. half the weight of the beam.

The center of gyration in the beam itſelf, without the weight W and P, is $\sqrt{\frac{2a^3}{ca}} = \sqrt{\frac{a^2}{3}}$ $= a\sqrt{\frac{1}{3}}$. (See Prob. XVI.)

$\sqrt{\frac{1}{3}} = \cdot 57735$, which × 10 = 5·7735, the diſtance of the center of gyration from the center of the beam.

Therefore, whether the whole maſs is diſperſed equally through a beam 20 feet long, whether it is collected in the center of gyration, or whether it is diffuſed through a circle, the radius of which is 5·7735 feet—yet a given power, applied at a given diſtance, would, in the ſame time, communicate the ſame angular velocity, or would turn it equally faſt round its axis.

Next, the reſiſtance of the beam muſt be reduced to the extremity of the arm, where the power acts, (ſee Prob. XIII.) which is done by multiplying the weight by the ſquare of the diſtance of the center of gyration, and dividing by the ſquare of the diſtance of the power P from the center of the beam. Thus the ſquare of the diſtance of the center of gyration is $\frac{a^2}{3}$, and the weight is $2m$; their product is $\frac{2ma^2}{3}$, which divided by a^2, gives $\frac{2m}{3}$ for the whole reſiſtance of the beam reduced to the end. And, as $2m$ = the whole weight, the reſiſtance which it affords to a power acting at the end is = $\frac{1}{3}$ of its weight.

We have therefore $W + P + \frac{2m}{3}$ = the maſs to be moved; and $100 - 60 = 40$ = the moving power. And the ſpace paſſed over in one ſecond, compared with gravity, is $\frac{P - W}{P + W + \frac{2m}{3}}$ $= \frac{100 - 60}{100 + 60 + 66.6} = .17647$, which multiplied by 16, gives 2.822 feet, the firſt ſecond.

Otherwiſe,

If we take 40, the moving power, from 100, we ſhall have each end loaded with 60lb. The center of gyration of the beam thus loaded will be expreſſed by $\left(\frac{2wa^2 + \frac{2ma^2}{3}}{2w + 2m}\right)^{\frac{1}{2}} = \sqrt{58.33}$: the maſs is $2w + 2m$, which multiplied by $\frac{2wa^2 + \frac{2ma^2}{3}}{2w + 2m}$ gives $2wa^2 + \frac{2ma^2}{3}$, which divided by a^2, the diſtance of the power, quotes $2w + \frac{2m}{3}$ for the whole reſiſtance reduced to the end of the beam, to which add $P - W$, and we have the whole maſs to be moved, by which divide the moving power, $P - W$, the quotient is the accelerating force, $= \frac{P - W}{2W + \frac{2m}{3} + P - W} = \frac{P - W}{\frac{2m}{3} + P + W} =$ $\frac{40}{120 + 66\frac{2}{3} + 40} = \frac{40}{226\frac{2}{3}} = .17647$, as before.

Otherwiſe,

If we divide the diſtance of the weight W by the diſtance of the center of oſcillation, we have the initial velocity.

If $2m =$ the whole weight of the beam; and
$2a =$ the length, as before.

Then will $\frac{2am + 3a\mathrm{W} + 3a\mathrm{P}}{3\mathrm{P} - 3\mathrm{W}}$ expreſs the diſtance of the center of oſcillation, by which divide a, the diſtance of W, and we have $\frac{3\mathrm{P} - 3\mathrm{W}}{2m + 3\mathrm{W} + 3\mathrm{P}} =$ .17647, the accelerating force compared with gravity, the ſame as before.

PROBLEM XXIX.

Given one end of a lever 6 feet, the other 10, the weight of the whole = 160lb. a weight W = 300lb. at the ſhorter end, and a power P = 500lb. at the longer end; required the accelerating force?

Let $a = 6$, the ſhorter end.
$b = 10$, the longer end.
$m = 160$, the whole weight.

Then, the ſquare of the diſtance of the center of gyration (Prob. XVI.) is $\frac{a^3 + b^3}{3a + 3b}$; which multiplied by the weight *m*, and divided by the ſquare of the diſtance of P, gives $\frac{a^3 + b^3}{3a + 3b} \times \frac{m}{b^2} =$ the whole reſiſtance of the lever at P. To this add the weight W, reduced to P, and the power P; and we have the whole reſiſtance at P = $\frac{a^3 + b^3}{3a + 3b} \times \frac{m}{b^2} + \frac{a^2 w}{b^2} + P = 648.533$.

Then, as 16, the whole length of the lever, is to 160, the whole weight, ſo is 10, the longer arm, to 100, its weight, which taken from 160, leaves 60, the weight of the ſhorter.

Then (Prob. XXI.) $\frac{2Wa + 60a - 100b}{2b} = 148$, which, if placed at P, would exactly balance the weight W; this, taken from P, leaves the moving power = 352, which divide by the whole maſs, and we have 648.533) 352 (.54276, which multiplied by 16, gives 8.684, the feet through which P would deſcend in one ſecond, if the direction was perpendicular.

Otherwiſe,

Let *m* = the weight of a foot in length.

Then, $\frac{mb^3 + 3Pb^2 + ma^3 + 3Wa^2}{mb^2 + 2Pb - ma^2 - 2Wa} \times \frac{2}{3} =$ the center of oſcillation $= \frac{608}{33}$; by which divide 10, the diſtance of P, and we have $\frac{330}{608} = .54276$; and as b is to a, ſo is the ſpace deſcribed by P to that deſcribed by W.

PROBLEM XXX.

Given, $a = 6$ feet, the ſhorter end of a lever; $b = 10$ feet, the longer end; $m =$ the weight of the longer end, $= 100$lb.; the weight of the ſhorter $= n = 60$; and let the power P be ſuſpended from the longer arm at a diſtance $d = 7$: required the accelerating force at P?

The center of oſcillation (ſee Prob. XXV.) will be $= \frac{2mb^2 + 6\text{P}d^2 + 2\text{P}a^2 + 6a^2\text{W}}{3mb + 6\text{P}d - 3na - 6a\text{W}}$; by which divide d, and the quotient is $\frac{3mbd + 6\text{P}d^2 - 3nad - 6a\text{W}d}{2mb^2 + 6\text{P}d^2 + 2na^2 + 6a^2\text{W}}$ $=$ the accelerating force at P; $= .359$ parts of gravity. Or, by the proceſs in Prob. XXVII. if $m =$ the weight of a foot, then $\frac{2\text{W}a + ma^2 - mb^2}{2d}$ $= 211\frac{6}{14}$, the weight which placed at P, would balance W: hence, the accelerating force, conſidered alone, is $500 - 211\frac{6}{14} = 288\frac{8}{14}$. The maſs to be moved, reduced to P, $= \frac{a^3 + b^3}{3a + 3b} \times \frac{m}{d^2}$ $+ \frac{a^2\text{W}}{d^2} + \text{P} = 803.129$, by which divide $288\frac{8}{14}$, and we have $.359$, as before.

PROBLEM

PROBLEM XXXI.

Given, the radius of a ſolid wheel, equally thick, $= d =$ 10 inches; the radius of the axis $= b =$ 1 inch; a power P $=$ 2lb. ſuſpended from the wheel; and a weight W $=$ 16lb. from the axis; the weight of the wheel $=$ 4lb. $= m$: required the ſpace deſcended through by the power compared with falling bodies?

Firſt, to find the center of gyration in the wheel.

Let $x =$ the diſtance of a particle from the center.
$q = .7854$.

Then will $4q = 2x =$ the circumference of a circle or ring, the diameter of which is $2x$.

The fluxion of its area will be $8qx\dot{x}$. And, As $8q\dot{x}x : 4qd^2$, the whole area, :: the weight of any particle a : the whole weight of the wheel, $= \frac{4aqd^2}{8qx\dot{x}} = m$; and if a, or its equal, $\frac{8mqx\dot{x}}{4qd^2}$, be multiplied by x^2, the ſquare of the diſtance of the particle, it gives $\frac{8mqx^3\dot{x}}{4qd^2} = \frac{2mx^4}{4d^2}$ (and when $x = d$) $= \frac{md^2}{2} = y^2m$; and $y^2 = \frac{d^2}{2}$

or $y = d\sqrt{\frac{1}{2}}$ = the diſtance of the center of gyration from the center of the wheel = 10 × .70716 = 7.0716 inches.

The weight of the wheel multiplied by the ſquare of the diſtance of the center of gyration, $(= \frac{d^2}{2} \times m)$ and divided by the ſquare of the radius, gives the reſiſtance which it affords at the circumference $= \frac{m}{2}$. The weight W, reduced to the circumference, is $\frac{wb^2}{d^2}$. Suppoſe the weight of the axis to be 5lb. = g, the reſiſtance at its ſurface will be $\frac{b^2}{2} \times g$, and at the circumference of the wheel $= \frac{gb^2}{2} \times \frac{b^2}{d^2}$, theſe added, give $\frac{gb^4}{2d^2} + \frac{m}{2} + \frac{wb^2}{d^2} + \mathrm{P}$, the whole reſiſtance; by which divide $\mathrm{P} - \frac{bw}{d}$, the moving power, (ſee Prob. XV.) and we have $\dfrac{\mathrm{P} - \frac{bw}{d}}{\frac{gb^4}{2d^2} + \frac{m}{2} + \frac{wb^2}{d^2} + \mathrm{P}} = \frac{.4}{4.1625}$ = .096, which multiplied by 16, gives 1.536 feet, the firſt ſecond.

If it is required to raiſe the weight W through a given ſpace in the leaſt poſſible time, by the action of the power P, it will be neceſſary to determine the diameter of the wheel to that of the axle. Let b = the diameter of the axle, and z that of the wheel, which is variable. The accelelerating force will be expreſſed by $\frac{2pz^2 - 2bwz}{2wb^2 + mz^2 + 2pz^2}$, = the ſpace through which P deſcends in 1 ſecond, compared with gravity, (1). And as the ſquare root of that ſpace is to 1 ſecond, the time in which it is deſcribed, ſo is the ſquare root of any other ſpace s, to $\sqrt{s} \times \sqrt{\frac{2wb^2 + mz^2 + 2pz^2}{2pz^2 - 2bwz}}$, the time of deſcribing it, which time is to be a minimum, or the leaſt poſſible: its fluxion is $\overline{2m\dot{z}z + 4pz\dot{z}} \times \overline{2pz^2 - 2bwz} - \overline{4p\dot{z}z - 2bw\dot{z}} \times \overline{2wb^2 + mz^2 + 2pz^2} = 0$. From which we get $mz^2 + 2pz^2 - 4pbz = 2wb^2$; and $z^2 - \frac{4pb}{2p+m} \times z = \frac{2wb^2}{2p+m}$: if $\frac{4pb}{2p+m} = n$, then $z = \sqrt{\frac{2wb^2}{2p+m} + \frac{n^2}{4}} + \frac{n}{2}$ = the radius of the wheel required. If the weight of the wheel is not conſidered, the Theorem becomes $\sqrt{\frac{wb^2}{p} + b^2} + b = z$.

If the power P is ſuſpended from the axle, and the weight from the wheel; and if the radius of the axle be 1, that of the wheel z, the weight of the wheel m, and its center of gyration y. Then will $\frac{Pz - W}{y^2m + Pz^2 + W} \times 16$ be the ſpace paſſed over by P in 1 ſecond; which

multiplied

multiplied by z, gives the ſpace paſſed over by W in the ſame time. Or if P is ſuſpended from the wheel, and W from the axle; then the ſpace paſſed over by P will be $\frac{pz^2 - wz}{y^2m + w + pz^2} \times 16$; which divided by z, gives the velocity of W.

On the maximum of machines, &c. at the end of a given time.

PROBLEM XXXII.

Given the weight of a beam 200lb. the arms of equal lengths, a weight of 100lb. ſuſpended from one end; it is required to aſcertain the weight ſuſpended from the other, ſo that when it is multiplied by its velocity the product ſhall be a maximum?

Put m = 200lb. the weight of the beam.
p = 100lb.
x = the weight ſought.

Then will $\frac{p - x}{\frac{m}{3} + p + x}$ (ſee Prob. XXVII.) be the accelerating force; and $\frac{px - x^2}{\frac{m}{3} + p + x}$ = the momentum of x, which in fluxions is $\overline{p\dot{x} - 2x\dot{x}} \times \overline{\frac{m}{3} + p + x} - \dot{x} \times \overline{px - x^2} = 0$; from which we get $x^2 + 2px + \frac{2mx}{3} = \frac{pm}{3} + p^2$, by putting the coefficients of x, $2p + \frac{2m}{3} = n$, and compleating the ſquare, we find $x = \sqrt{\frac{pm}{3} + p^2 + \frac{n^2}{4}} - \frac{n}{2} = 44.152$.

$\frac{p - x}{\frac{m}{3} + p + x}$, the ſpace paſſed over in the firſt ſecond, $= 2.63$, which multiplied by the weight x, gives the product 11.7; which product is greater than can be obtained by aſſuming x either more or leſs than given by the theorem. But if the weight of the beam is altered, although the power p remains the ſame, the weight x will be found to vary. For example, if the weight of the beam is 30, then x (from the above theorem) will be found $= 42$, and its velocity .381, which × 42, gives 16, the momentum. So that by diminiſhing the weight of the beam, the ſame power, in an equal time, produces a greater effect on x; not that the abſolute effect is greater, but the additional force which is now exerted on x was before employed to move the beam.

Again, let the weight of the beam be 360, $p = 100$ as before, and we ſhall find, as above, the value of $x = 45.33$, its velocity $= .206$, and their product, or the momentum, $= 9.337$.

Therefore,

Therefore, when the effect is the greateſt, the weight x increaſes as the beam becomes heavier; but the velocity diminiſhes in a greater ratio: of conſequence, the effect produced by the power p, in raiſing weights, is diminiſhed by increaſing the weight of the machine.

PROBLEM XXXIII.

Given the length and weight of a beam, the power ſuſpended from the longer arm, to determine the weight at the end of the ſhorter, when the effect is a maximum?

Put a = length of the ſhorter = 6.
n = its weight = 60.
b = the longer arm = 10.
m = its weight = 100.
p = 500.

Then will $\frac{amb + 2apb - na^2 - 2a^2w}{mb^2 + 3pb^2 + na^2 + 3a^2w} \times \frac{3}{2} =$ the accelerating force at w. Let the known terms in the numerator be put = N, and thoſe in the denominator = Q, and we ſhall have $\frac{N - 2a^2w}{Q + 3a^2w} \times w$ a maximum; in fluxions $\overline{N\dot{w} - 4a^2w\dot{w}} \times \overline{Q + 3a^2w} - 3a^2\dot{w} \times \overline{Nw - 2a^2w^2} = 0$.

And $w = \frac{1}{a^2}\sqrt{\frac{QN}{6} + \frac{Q^2}{9}} - \frac{Q}{3a^2}$.

$amb + 2apb - na^2 = N = 63840.$

$mb^2 + 3pb^2 + na^2 = Q = 162160.$

$\frac{QN}{6} = 1725382400.$

$\frac{Q^2}{9} = \underline{2921762844.44}$

Sum 4647145244.44, the ſquare root of this is 68169.9, which multiplied by $\frac{1}{a}$ $(= \frac{1}{36})$ is 1893.61, from which ſubtract 1501.4814 $(\frac{Q}{3a^2})$ leaves $392.13 = w$, the weight required.

The ſame concluſion may be brought out by a different proceſs. If $\sqrt{\frac{a^3 + b^3}{3a + 3b}} = r$, the center of gyration in the beam. Then will

$$\frac{P + \frac{m}{2} - \frac{an}{2b} - \frac{aw}{b}}{\frac{m+n}{b^2} \times r^2 + P + \frac{a^2w}{b^2}} = \text{the velocity of } P.$$

Then let $P + \frac{m}{2} - \frac{an}{2b} = N$; and $\frac{m+n}{b^2} \times r^2 + P = Q$.

And we have $\frac{Nw - \frac{aw^2}{b}}{Q + \frac{a^2w}{b^2}} =$ the effect; which by the Problem muſt be the greateſt poſſible.

The fluxion of this expreſſion is $\overline{N\dot{w} - \frac{2aw\dot{w}}{b}}$ $\times$

$\times \overline{Q + \frac{a^2w}{b^2}} - \frac{aw}{b^2} \times \overline{Nw - \frac{aw^2}{b}} = 0$. Whence

$$w = \frac{b}{a}\sqrt{\frac{NQb}{a} + \frac{Q^2b^2}{a^2}} - \frac{Qb^2}{a^2}.$$

The value of N and Q in this proceſs will be different from what they are found in the laſt; but the value of w will be exactly the ſame.

For example,

$$P + \frac{m}{2} - \frac{an}{2b} = N = 532.$$

$$\frac{m+n}{b^2} \times r^2 + P = Q = 540.533.$$

$$\frac{NQb}{a} = 479272.88.$$

$$\frac{Q^2b^2}{a^2} = 811600.79.$$

Sum $\overline{1290873.67}$.

$\sqrt{1290873.67} = 1136.16$; which multiplied $\frac{b}{a}$, gives 1893.6

Subtract $1501.4 = \frac{Qb^2}{a}$

Rem. $392.2 = w$, the anſwer as before.

PROBLEM XXXIV.

Let the weight P be ſuſpended from the arm, at any diſtance from the end; required the weight of w, in caſe of a maximum?

Let a = the ſhorter arm.
b = the longer.
d = the diſtance of P from the center.
n = the weight of the ſhorter end.
m = the weight of the longer.

The center of oſcillation of the whole, will be expreſſed by $\frac{2mb^2 + 6pd^2 + 2na^2 + 6a^2w}{3mb + 6pd - 3na - 6aw}$, by which divide a, the diſtance of the weight w, and we have $\frac{3amb + 6apd - 3na^2 - 6a^2w}{2mb^2 + 6pd^2 + 2na^2 + 6a^2w}$ = the accelerating force, or initial velocity of w; which, when multiplied by w, muſt be a maximum: its fluxion is $\overline{amb\dot{w} + 2apd\dot{w} - na^2\dot{w} - 4a^2w\dot{w}} \times \overline{mb^2 + 3pd^2 + na^2 + 3a^2w} - 3a^2\dot{w} \times \overline{ambw + 2apdw - na^2w - 2a^2w^2} = 0$, which multiplied and reduced, gives $6a^4w^2 + a^4nw + 3na^3w + 12a^2pd^2w + 4ma^2b^2w = am^2b^3 + 2apdmb^2 + 3ambpd^2 + 6ap^2d^3 + a^3mbn + 2a^3pdn - nma^2b^2 - 3na^2pd^2 - n^2a^2$.

Let the coefficients of the firſt power of w, $\frac{n}{6} + \frac{n}{2a} + \frac{2pd^2}{a^2} + \frac{2mb^2}{3a^2}$, $= c$; then, by compleating the ſquare, &c, we get $w = \frac{1}{a^2}\sqrt{\frac{am^2b^3 + 2apdmb^2 + 3ambpd^2 + 6ap^2d^3 + a^3mbn + 2a^3pdn - nma^2b^2 - 3na^2pd^2 - n^2a^2}{6} + \frac{c^2}{4}} - \frac{c}{2}$.

EXAMPLE

EXAMPLE.

Let $a = 1$, $b = 5$, $d = 3$, $m = 5$, $n = 1$, $p = 10$.

Firſt, to find the value of c, the coefficient of w.

$\frac{n}{6}$ = .166
$\frac{n}{2a}$ = .5
$\frac{2pd^2}{a^2}$ = 180.
$\frac{2mb}{3a^2}$ = 83.33

Sum 264 = c.

Under the venculum we have,

Poſitive terms.		*Negative terms.*	
am^2b^3 =	3125	nma^2b^2 =	125
$2apdmb^2$ =	7500	$3a^2npd^2$ =	270
$3ambpd^2$ =	6750	n^2a^2 =	1
$6ap^2d^3$ =	16200		
a^2mbn =	25	Sum	396
$2a^2pdn$ =	60		

Sum poſitive 33660
Sum negative 396

Difference 33264 ÷ 6 = 5544, to which add 17424, ($\frac{c^2}{4}$) and extract the ſquare root of the ſum, which is found $= 151.552$; this divided by a^2 (1), and $\frac{c}{2}$ taken from it, leaves the value of $w = 19.55$, the weight required.

The above Theorem may be contracted by putting the known quantities, or terms, in the numerator, $amb + 2apd - na^2, = \text{N} = 84$; and thoſe in the denominator, $mb^2 + 3pd^2 + na^2, = \text{Q} = 396$. Then, by the Theorem in the laſt Problem, $\left(\frac{1}{a^2}\sqrt{\frac{\text{QN}}{6} + \frac{\text{Q}^2}{9}} - \frac{\text{Q}}{3a^2}\right)$ we have the ſolution.

$$\frac{\text{QN}}{6} = 5544$$

$$\frac{\text{Q}^2}{9} = \underline{17424}$$

their ſum 22968, the ſquare root of which $\div a^2$, or $\times \frac{1}{a^2}$, is 151.55; and $\frac{\text{Q}}{3a^2} = 132$, which taken from 151.55, leaves 19.55 as before, but by a much eaſier proceſs.

On the maximum of bodies, when the ſpace is given.

I have now, by various proceſſes, by different methods, and from different principles, demonſtrated what the greateſt effect is, which a given body, acted upon by gravity, can produce in a given *time*. I ſhall now proceed to demonſtrate what the greateſt effect is, which a given body can produce, in falling through a given *ſpace*. If the ratio of w to x is ſuch that the effect produced is the greateſt at the end of one ſecond of time, it will be the greateſt at the end of any other period of time. But if the ſpace is given when the effect is greateſt, the ratio of w to x will vary, or be different to what it was when the time was given.

PROBLEM XXXV.

It is required to determine the greateſt effect that a given body can produce, in falling through a given ſpace?

Let $w =$ the weight of the body, or power.
$x =$ the reſiſtance, or weight to be raiſed.
And let then be connected by a line paſſing over a pulley. (See Fig. 11.)

The accelerating force will be $\frac{w - x}{w + x}$ (Prob. 1) which multiplied by 16, gives the feet paſſed over in the firſt ſecond. But the time and velocity are as the ſquare root of the ſpace, and will be expreſſed by $\sqrt{\frac{w - x}{w + x}}$, which multiplied by x, gives $x\sqrt{\frac{w - x}{w + x}}$; which, by the Problem, muſt be a maximum. Its fluxion is $\overline{2wx\dot{x} - 3x^2\dot{x}}$

×

$\times \overline{w + x} - x \times \overline{wx^2 - x^3} = 0$. From which we have $2x^3 + 2wx^2 = 2w^2x$; which divided by $2x$, gives $x^2 + wx = w^2$; and $x^2 + wx + \frac{w^2}{4} = w^2 + \frac{w^2}{4}$. By extracting the root on both ſides, we get $x = \frac{w}{2}\sqrt{5} - \frac{w}{2}$.

EXAMPLE.

Given $w = 10$; required x?

The ſquare root of $5 = 2.236076$, which multiplied by $\frac{w}{2}$ $(= 5)$, gives 11.18038, from which ſubtract $\frac{w}{2}$, and there remains $6.18038 = x$. When $w = 10$, *this number, or value of* x, *is the maximum for any ſpace, as* 4.142 *is for any time.*

Note 1. It appears by Prob. III, the time being given, that, when the effect is greateſt, the power deſcends through 6.627 feet in one ſecond, and, of conſequence, from the laws of falling bodies, through 26.5 in two ſeconds, and through $3 \times 3 \times 6.627 = 59.6$ in three ſeconds, &c. Therefore, the weight remaining the ſame, the ſpace through which it is raiſed by a given power will be as the ſquare of the time.

Note 2. It appears by this Problem, the ſpace being given, that there is a limited time, in which, the power will produce the greateſt effect.

Note 3. If T = any given time, the number of ſtrokes made in that time will be $\frac{\mathrm{T}}{\sqrt{\frac{sw+sx}{rw-rx}}} = \mathrm{T}\sqrt{\frac{rw-rx}{sw+sx}}$, which multiplied by x, gives $\mathrm{T}\sqrt{\frac{rwx^2-rx^3}{sw+sx}}$ for the maximum; the fluxion of which is the ſame as that already found in this Problem.

EXAMPLE.

Let $\mathrm{T} = 3600$ ſeconds, or one hour.
$w = 10.$
$x = 4.142.$
$r = 16.$
$s = 26.5.$

Then $\sqrt{\frac{sw+sx}{rw-rx}} = 2$ ſeconds, the time of one ſtroke; and $\mathrm{T}\sqrt{\frac{rw-rx}{sw+sx}} = 1800$, the ſtrokes in one hour; which multiplied by 4.142, the weight raiſed at one ſtroke, gives 7456, the weight raiſed in one hour.

But if $x = 6$.

Then $\sqrt{\frac{sw + sx}{rw - rx}} = 2.57$ ſeconds, the time of one ſtroke, by which divide 3600, and we get 1400.7, the number of ſtrokes in one hour; which multiplied by 6, the weight raiſed by one ſtroke, gives 8404.2 for the whole weight raiſed in one hour.

PROBLEM XXXVI.

Given a power w applied to the circumference of a wheel; required the weight x ſuſpended from the axle, when the product ariſing from the weight into the velocity is the greateſt poſſible at the end of a given ſpace?

Put a = radius of the axle.
b = radius of the wheel.
s = the given ſpace through which w deſcends = 25.
r = 16.
x = weight ſought.

The accelerating force at w will be $\frac{b^2rw - baxr}{b^2w + a^2x}$. And as $\sqrt{\frac{b^2rw - baxr}{b^2w + a^2x}} : 1'' :: \sqrt{s} : \sqrt{\frac{swb^2 + sa^2x}{rwb^2 - rabx}}$, the time in which w falls through the ſpace s; by which divide the time T, and we have $T\sqrt{\frac{rwb^2 - abrx}{swb^2 + a^2sx}}$ = the number of ſtrokes made in the ſaid time. The number of ſtrokes, multiplied by the weight x, which is raiſed at one ſtroke, gives $\mathrm{T}\frac{\sqrt{abwx^2 - a^2x^3}}{\sqrt{swb^2 + sa^2x}}$ for the maximum. Its fluxion is $2bwx\dot{x} - 3ax^2x \times \overline{b^2w + a^2x} - a^2\dot{x} \times \overline{bwx^2 - ax^3} = 0$. From which, by multiplication and reduction, we get $2a^2x^2 + 3b^2wx - abwx = \frac{2b^3w^2}{a}$; and $x^2 + \frac{3b^2w - abw}{2a^2} \times x = \frac{b^3w^2}{a^3}$. Let $\frac{3b^2w - abw}{2a^2} = n$: then, by compleating the ſquare, &c. we get $x = \sqrt{\frac{b^3w^2}{a^3} + \frac{n^2}{4}} - \frac{n}{2}$.

EXAMPLE.

Let $a = 1$, $b = 2$, $w = 10$.

Then $x = 12.74$, and is raiſed 12.5 feet, while w falls 25 feet. The time of deſcent is expreſſed by $\sqrt{\frac{wb^2 + sa^2x}{rb^2w - baxr}}$, and in this caſe is 2.38 ſeconds.

PROBLEM XXXVII.

Let the weight of the wheel be given, $= m$; the center of gyration y; a power p acting at the circumference: required the weight w ſuſpended from the axle, when the effect of p in raiſing it through a given ſpace is the greateſt poſſible?

Let the radius of the axle be $= 1$; that of the wheel $= r$.

Then will $\frac{pr^2 - wr}{y^2m + pr^2 + w} \times 16 =$ the number of feet which w aſcends in one ſecond. And as $\sqrt{\frac{pr^2 - wr}{y^2m + pr^2 + w}} : 1'' :: \sqrt{s} : \sqrt{\frac{y^2m + pr^2 + w}{pr - w}} \times \overline{\frac{s}{16}}\Big|^{\frac{1}{2}}$, the time of aſcent through the ſpace s; by which divide the time T, and the quotient is the number of ſtrokes made in that time; by which multiply w, the weight raiſed at one time, and we have the whole weight raiſed in any given time T, and which by the Problem muſt be the greateſt poſſible: hence we have $\sqrt{\frac{prw^2 - w^3}{y^2m + pr^2 + w}}$ for the maximum. The fluxion of which is $\overline{2prw\dot{w} - 3w^2\dot{w}} \times \overline{y^2m + pr^2 + w} - \dot{w} \times \overline{prw^2 - w^3} = 0$. From which we get $w^2 + \frac{3pr^2 + 3my^2 - pr}{2} \times w = prmy^2 + p^2r^2$; and by putting $n =$ the coefficients of w, we get $w = \sqrt{p^2r^3 + prmy^2 + \frac{n^2}{4}} - \frac{n}{2}$.

EXAMPLE I.

Let $p = 10$, $r = 2$, $y^2 = 2$, $m = 1$.

Then $\frac{3pr^2 + 3my^2 - pr}{2} = n = 53$; and $w = \sqrt{800 + 40 + 702.25} - 26.5 = 12.77$.

EXAMPLE II.

Let the weight of the wheel be 4, $= m$, and $y^2 = 6$, to find w?

$n = 86$; and $w = \sqrt{800 + 480 + 1849} - 43 = 12.9$; and the time of riſing 25 feet will be 4 ſeconds.

EXAMPLE III.

Given the weight of the wheel 10lb.; the power p 10lb.; $y^2 = 10$: required w, and the time of riſing 25 feet?

From the above data we find $n = 200$, and $w = \sqrt{800 + 2000 + 10000} - 100 = 13.13$; and the time is $\sqrt{\frac{my^2 + pr^2 + w}{pr - w}} \times \sqrt{\frac{25}{16}} = 5.9$ ſeconds.

It appears that, by increaſing the weight of the wheel, &c. the time of aſcent is prolonged near 2 ſeconds, and the weight raiſed is very little more. (See Prob. XXXII.)

PROBLEM XXXVIII.

Let the power w be applied to the axle, to raiſe the weight x ſuſpended from the wheel (the reſt of the notation as in the laſt Problem) through a given ſpace; required x, when the effect is the greateſt?

$\frac{w - rx}{y^2m + r^2x + w}$ is the ſpace paſſed over by w compared with gravity (1); and, by the laſt Problem, $\frac{wx^2 - rx^3}{y^2m + w + r^2x}$ is the greateſt effect.

Its fluxion is $\overline{2wx\dot{x} - 2rx^2\dot{x}} \times \overline{y^2m + w + r^2x} - r^2\dot{x} \times \overline{wx^2 - rx^3} = 0$. From which we find $x = \sqrt{\frac{2w^2 + 2mwy^2}{2r^3} + \frac{n^2}{4}} - \frac{n}{2}$.

Note, $\frac{3rw + 3my^2 - r^2w}{2r^3} = n$.

Note, In conſtructing machines, it will, in general, be of more importance to compute the effect produced by a body in paſſing over a given ſpace, than at the end of a given time.

But ſuppoſe it ſhould be required to aſſign the diſtance, at which a given power muſt be applied from the axis of a wheel, ſo as to produce a greater number of revolutions, in falling through a given ſpace, than could be produced by any different application of the ſaid power, in falling through the ſame ſpace, in any proportionable part of the time. The anſwer is, apply the power as near the axis as ſtrength in the machine and other circumſtances will admit. (See Prob. XIV and XV.) It has been proved, that if a body falls through a given ſpace, ſuppoſe 20 feet, in 20 ſeconds, and produces 100 revolutions, if applied at twice the diſtance from the axis it will fall through the ſame ſpace in 10 ſeconds, but will only produce 50 revolutions, the *vis inertiæ* of the power not conſidered; but if it is taken into the computation, it will require more than 10 ſeconds to make 50 revolutions.

For example,

Let m = the weight of the wheel = 31.9.
r = the center of gyration = 10.
p = the power = 10.
x = the diſtance from the axis at which p is applied = 1.

Then $\frac{16px^2}{mr^2+px^2}$ = the feet fallen through by p in the firſt ſecond = $\frac{1}{20}$. And as $\frac{1}{20}$ foot : $1^{2''}$ :: 20 feet : $20^{2''}$, the time of deſcent. But if $x = 2$, by the ſame proceſs, the time of deſcent will be found = 10.046 ſeconds; and in falling through a given ſpace, the number of revolutions muſt be inverſely as the diameters. Hence, by increaſing the diſtance at which the power is applied, we diminiſh the effect, when the ſpace is given. But if the time was given, the diſtance might be found ſuch, that the effect would be the greateſt.

PROBLEM XXXIX.

Given the weight of a fly $= m = 31.9$; the center of gyration $= r = 10$; the power $= p = 10$: required x, the diſtance at which p muſt be applied, to produce, in a given time, the greateſt number of turns?

$\frac{mr^2}{x^2} =$ the whole reſiſtance of the wheel reduced to p, to which if we add p, we have $\frac{mr^2}{x} + p =$ the whole maſs; the moving power being divided by this, gives $\frac{px^2}{mr^2 + px^2}$, which multiplied by 16, gives the ſpace paſſed over by p in the firſt ſecond; which divided by $2x \times q$ $(2x \times 3.1416)$, gives $\frac{px}{2q \times \overline{mr^2 + px^2}}$, the number of revolutions performed in that time. The fluxion of which is $p\dot{x} \times \overline{mr^2 + px^2} - 2px\dot{x} \times px = 0$. From which we get $pmr^2 + p^2x^2 = 2p^2x^2$; and $x = r\sqrt{\frac{m}{p}} = 10\sqrt{\frac{31.9}{10}} = 17.8$, which, in this caſe, is the diſtance at which the power ought to be applied, in order to produce the greateſt number of revolutions in a given time.

PROBLEM XL.

Given the weight of a beam ſuſpended by the middle, $= m = 200$lb.; and a weight $p = 100$, ſuſpended from one end; required the weight to be ſuſpended from the other, ſo that the effect may be the greateſt after having aſcended through a given ſpace?

It is preſumed that the ends of the beam are circular, and that the weights move in perpendicular directions.

The accelerating force will be $\frac{p-x}{\frac{m}{3}+p+x}$ (ſee Prob. XXVII and XXXII); the ſquare root of which multiplied by x, gives the maximum $= \frac{px^2-x^3}{\frac{m}{3}+p+x}$. Its fluxion is $\overline{2px\dot{x}-3x^2\dot{x}} \times \overline{\frac{m}{3}+p+x} - \dot{x} \times \overline{px^2-x^3} = 0$. From which we have $x^2+px+\frac{mx}{2}=\frac{pm}{3}+p^2$. Let $p+\frac{m}{2}=n$; then, by compleating the ſquare, &c. we find $x = \sqrt{\frac{pm}{3}+p^2+\frac{n^2}{4}}-\frac{n}{2}$.

On

On Steam-Engines.

PROBLEM XLI.

IT has been demonſtrated, that when the preſſure upon the ſteam-piſton is to the weight of the pump-rods and water as 1000 to 618, the effect is the greateſt when the piſton deſcends. And if it is ſo adjuſted, that the weight of the rods, &c. when in the water, be to the weight of the ſteam-piſton, plug-frame, &c. as 1000 to 618, the engine will do the moſt work in a given time. If the preſſure of the air is taken at 14lb. upon an inch, then as 1000 : 618 :: 14 : 8.65, the weight with which the end over the ſhaft ſhould be loaded for every ſquare inch of the ſteam-piſton; (not conſidering the friction of the parts, or weight of the beam), which would amount to 1245lb. for every ſquare foot of the ſteam-piſton: but, every thing conſidered, this is far too much, and we may, in general, make the load from 800 to 860lb. per foot. In large engines, the friction is leſs, in proportion to their power, than in ſmall engines, of conſequence the load may be more.

Put d = diameter of the ſteam-cylinder, in feet.
$q = .7854$.
l = the length of the ſtroke, in feet.
p = the preſſure per foot, in pounds avoirdupoiſe.

$w =$

Put $w = 62.5$lb. weight of a cubic foot of water.
h = the number of feet the water is to be raiſed.
c = diameter of the pump, in feet.
$g = 6.1276$, ale gallons in a cubic foot.
$n = 10.28$, cubic feet in a hogſhead.

Then will

qd^2 = area of the ſteam piſton, in feet.

pqd^2 = the weight of the water to be lifted, in pounds avoirdupoiſe.

$\frac{pqd^2}{w}$ = cubic feet of water in the pump; which divided by the height h, gives

$\frac{pqd^2}{hw}$ = the area of the pump-piſton; which multiplied by the length of the ſtroke, l, gives

$\frac{lpqd^2}{hw}$ = cubic feet raiſed per ſtroke.

From which equations we have the following Theorems.

Theo. 1. $d = \sqrt{\frac{whc^2}{p}}$.

Theo. 2. $c = \sqrt{\frac{pd^2}{hw}}$.

Theo. 3. $h = \frac{pd^2}{wc^2}$.

Theo. 4. lqc^2 = cubic feet per ſtroke.

Theo. 5. $glqc^2$ = gallons per ſtroke.

EXAMPLE I.

Given, the diameter of the ſteam-cylinder, 3 feet; the length of the ſtroke, 6 feet; the depth of the well, 50 feet: required the diameter of the pump, and the quantity of water raiſed at one ſtroke?

Here, we have given, the diameter of the cylinder $= 3 = d$; the length of the ſtroke $= 6 = l$; the depth of the well $= 50 = h$; to find c: which by Theo. 2, is $= \sqrt{\frac{pd^2}{hw}} = \sqrt{\frac{7515}{3125}} =$ 1.55 feet. And the quantity per ſtroke is, by Theo. 4, $= lqc^2, = 6 \times .7854 \times 2.404 =$ 11.328 cubic feet.

In the above, p is taken at 835lb. inſtead of 1245, which it ought to be taken at, were it not for the various ſorts of reſiſtance, friction, &c. which do not enter into the above computation.

If all cylinders were equally well bored, and the work executed in the ſame manner, an eſtimate might be made for friction, &c. which compared with the moving power, would always be, in the cylinder, as the circumference to the area of the piſton: or, in different engines, it would be as the diameters of the cylinders and pumps. To which ſhould be added, the weight of the beam, rods, plug-frame, &c.

EXAMPLE II.

Given, the diameter of the pump, 1 foot; the height to which the water is to be raiſed, 240 feet; to find the diameter of the ſteam-cylinder?

Here we have $w = 62.5$lb.; $h = 240$; $c = 1$: and $whqc^2 = pqd^2 =$ the whole weight of the cylinder of water to be lifted. From which we find, Theo. 1, $d = \sqrt{\frac{whc^2}{p}} = \sqrt{\frac{15000}{835}} = \sqrt{17.96} = 4.23$ feet, the diameter required.

EXAMPLE III.

Given, the quantity of water to be raiſed in one hour, 200 hogſheads; the depth of the water, 100 yards; length of the ſtroke, 6 feet; number of ſtrokes in a minute, 10: required the diameter of the cylinder and pump?

200 hogſheads, the quantity raiſed in one hour, is equal to 12600 gallons, which divided by 600, the number of ſtrokes made in the ſame time, quotes 21, the gallons to be raiſed at one ſtroke. Therefore, (Theo. 5) $21 = glqc^2$, and $c = \sqrt{\frac{21}{glq}} = .852$ feet, or 10.22 inches, the

diameter of the pump: and (per Theo. 1.) $d = \sqrt{\frac{whc^2}{p}} = 4.04$ feet, the diameter of the cylinder.

N. B. If the weight of the pump-rods and plug-frame be taken into the computation; let their weight, when the weight of the ſteam-piſton is taken from them, be put $= n$ pounds.

Then will $d = \sqrt{\frac{whc^2}{p} + \frac{n}{qp}}$. And $c = \sqrt{\frac{pd^2}{hw} - \frac{n}{qhw}}$.

EXAMPLE IV.

Required the dimenſions of an engine to raiſe 250 hogſheads per hour to the altitude of 55 yards, the weight of the rods being 2000lb.?

250 × 63 = 15750, the gallons per hour, which, at 12 ſtrokes in a minute, is 22 gallons per ſtroke = 3.59 cubic feet $= qlc^2$, and $c^2 = \frac{3.59}{ql}$; the ſquare root of which is .87 feet, or 10.4 inches, the diameter of the pump. And $d = \sqrt{\frac{whc^2}{p} + \frac{n}{pq}} = 3.53$ feet, the diameter of the cylinder.

PROBLEM XLII.

Given, the length of an engine beam $= 2a$, its weight $= m$, weight of each horſe head, or arch at the end, $= c$, weight of the ſteam-piſton and chain $= s$, weight of the plug-frame and its arch $= u$, its diſtance from the center $= t$, preſſure of the atmoſphere upon the piſton $= p$; required the weight of water lifted, or reſiſtance overcome, when the effect is greateſt?

Suppoſe the beam to be ſuſpended by the center, and that the piſton and chain are balanced by a part of the pump-rod and chain, at the other end. Then will the whole weight of the beam, arches, piſton, &c. be $2c + 2s + m$, which put $= r$, and let the center of gyration in the beam be y, $= \sqrt{\frac{6ca^2 + 6sa^2 + ma^2}{6 \times \overline{c + s} + 3m}} = \sqrt{\frac{a^2}{3} \times \overline{\frac{4c + 4s}{2c + 2s + m} + 1}}$, then will $\frac{ry^2}{a^2}$ be the whole reſiſtance of the beam, &c. reduced to the end to which the power is applied.

Let $x =$ the reſiſtance ſought. Then will $\frac{\overline{p - x} \times 16}{p + \frac{ry^2}{a^2} + x}$ be the ſpace paſſed over in the firſt ſecond.

Let $l =$ the length of the ſtroke. Then, As $\frac{\overline{p - x} \times 16}{p + \frac{ry^2}{a^2} + x} : 1^{2\prime\prime} :: l : l \times \frac{p + \frac{ry^2}{a^2} + x}{\overline{p - x} \times 16}$; the ſquare root of which, is the time of paſſing over the ſpace l. And the greateſt effect will be expreſſed by $\sqrt{\frac{px^2 - x^3}{p + \frac{ry^2}{a} + x}}$; the fluxion of which is $\overline{2px\dot{x} - 3x^2\dot{x}} \times \overline{p + \frac{ry^2}{a^2} + x} - \dot{x} \times \overline{px^2 - x^3} = 0$. From which we get $x^2 + \overline{p + \frac{3ry^2}{2a^2}} \times x = p^2 + \frac{pry^2}{a^2}$. Let $p + \frac{3ry^2}{2a^2} = n$, the coefficients of x; and we ſhall have $x^2 + nx + \frac{n^2}{4} = \frac{pry^2}{a^2} + p^2 + \frac{n^2}{4}$: from which we find $x = \sqrt{\frac{pry^2}{a^2} + p^2 + \frac{n^2}{4}} - \frac{n}{2}$. Then, to find the additional weight of the pump-rod, ſo that the aſcent of the piſton may be performed in the ſame time as the deſcent, we have As $x : p :: s : \frac{sp}{x} =$ the whole weight of the pump-rods and chain, from which ſubtract s, and we have the additional weight ſought, which, if the rods be not heavy enough, muſt be ſupplied by an equivalent weight added to the end of the beam, and muſt be further increaſed by reducing the plug-frame to the end of the beam, in the following manner, viz. $\frac{t^2u}{a^2} =$ the reſiſtance at the end of the beam, and As $x : p :: \frac{ut^2}{a^2} : \frac{put^2}{xa^2}$; hence $\frac{sp}{x} + \frac{put^2}{xa^2} =$ the whole weight ſuſpended at the pit end of the beam. This diminiſhed by s, and the remainder taken from x, leaves $x - \frac{sp - sx}{x} + \frac{put^2}{xa^2} =$ the weight of water to be lifted, not conſidering the friction.

EXAMPLE I.

Let $a = 12$, $m = 1000$, $c = 280$, $s = 700$, $u = 200$, $t = 7$, $p = 2000$. Then will $r = 2960$, and $y^2 = 111.55$; alſo $\frac{ry^2}{a^2} = 2293$; $n = 5440$.

$$\frac{pry^2}{a^2} = 4586000$$
$$p^2 = 4000000$$
$$\frac{n^2}{4} = \underline{7398400}$$

Sum 15984400, the ſquare root of this is 3998, from which ſubtract $\frac{n}{2}$ ($= 2720$), and we have $1278 = x$, the whole reſiſtance, which muſt be diminiſhed by $\overline{\frac{sp}{x} - s} + \frac{put^2}{xa^2} = 502$: and we have 785 for the weight of the water to be lifted. And if $l = 6$ feet, the time of deſcending will be $\left(\sqrt{l \times \frac{p + \frac{ry^2}{a^2} + x}{\overline{p - x} \times 16}}\right)$ 1.6 ſeconds, which doubled, gives the time in which the piſton, &c. aſcends and deſcends = 3.2 ſeconds, which would give 18 ſtrokes in one minute, if the piſton made no ſtop at the top or bottom.

EXAMPLE II.

Let $a = 12$, $m = 5000$, $c = 1000$, $s = 2000$, $u = 500$, $t = 7$, $p = 14250$. Then $2c + 2s +$

$m =$

$m = 11000 = r$; alſo $\frac{a^2}{3} \times \overline{\frac{4c+4s}{2c+2s+m}+1} =$ $100.3636 = y^2$; and $\frac{ry^2}{a^2} = 7666\frac{2}{3}$. $\frac{3ry^2}{2a^2}+p =$ $25750 = n$: from which we have

$$\frac{pry^2}{a} = 109250000$$
$$p^2 = 203062500$$
$$\frac{n^2}{4} = \underline{165765625}$$

Sum 478078125, its ſquare root is 21865, from which take $\frac{n}{2}$, and we have $9990 = x$; from which take $\frac{sp}{x} - x + \frac{pux^2}{ma^2}$ (= 1095), and we have the weight of water = 8895lb.

In the firſt Example, every ſquare foot of the piſton raiſes 1288lb.; and in the ſecond, 1413lb. In the firſt, the ſtrokes per minute are 18.75; in the ſecond, 17.96; which multiplied by the weight raiſed at one ſtroke, gives 25377 and 24150, which difference ariſes from the ſuperior weight of the beam, &c. in the ſecond Example.

Had the value of x been ſought, as in Prob. XXXV, neglecting the weight of the beam, &c. in the firſt Example it would be found 1236, and the ſtrokes per minute would be 24, hence the weight raiſed per minute = 29664lb.; in the ſecond Example, the weight per ſtroke, and the ſtrokes per minute, would be the ſame as in the firſt. Hence, computations on the power of engines, or machines of any kind, muſt be erroneous when the weight or reſiſtance of the machine is neglected.

When an engine is employed to raiſe a large quantity of water to a little height, the ſpear, or pump-rod, is not heavy enough to lift the piſton with a due velocity; hence, it has been common to load that end of the beam: but it would certainly be an advantage to make the end over the pump ſomething longer, ſo that by its increaſed weight it may bring up the piſton with a proper velocity, and alſo make a longer ſtroke in the pump, and of conſequence bring up a greater quantity of water (ſee Prob. XIX, XX, and XXI). The theory conſiders every part of an engine perfect, but as this can ſcarcely ever be true, it may be an advantage to make the load leſs than the theory aſſigns, in order that it may ſtrike faſter, by which means the loſs of water by the piſton, through the valves, &c. will in part be diminiſhed. And if the method of computing the power in the laſt Problem ſhould be thought too intricate, an engine conſtructed by the Theorems in Prob. XLI. will, if well executed, be pretty near the maximum.

A TREATISE

A TREATISE ON MILLS.

PART III.

ON THE VELOCITY OF FLOWING WATER.

To determine the quantity of water diſcharged in a given time, through given apertures, and at different depths below the ſurface, is, in many caſes, of great importance. Sir Iſaac Newton, in his *Principia*, Book 2, Theo. 8, Prob. 36, has demonſtrated that the velocity of water, flowing through holes in the bottom or ſide of a veſſel, ought to be equal to the velocity which a heavy body would acquire in falling through a ſpace equal to the diſtance between the ſurface of the water and the place where it is diſcharged. Hence, at the depth of 16 feet, a ſtream of 32 feet in length ought to flow out in one ſecond of time. And from the laws of falling bodies, it follows, that as the ſquare root of 16, is to the velocity of the ſtream flowing out at that depth; ſo is the ſquare root of any other depth, to the velocity at that depth. That is, the expence of water through equal apertures, is directly as the ſquare roots of their depths. But Sir Iſaac, in making experiments, found the velocity thus determined to be too great, which in different caſes he corrected. The friction againſt the ſides of the hole, and the oblique direction of the particles of water before they reach the hole, both tend to diminiſh the velocity of the ſtream. And if theſe cauſes could be removed, eſpecially the latter, the *Newtonian theory* would certainly be confirmed by experiment; or rather, experiment would exactly agree with theory. If we ſuppoſe water running into the top of a tube of equal diameter, and that there is no attraction, or friction, between the particles of water and the ſaid tube, the velocity of the water, or of each particle at the bottom, will be the ſame, or equal to that which they would have acquired in falling through the ſame ſpace, without the tube, toward the earth. Hence to obtain the true velocity, under different circumſtances, we muſt correct the computed velocity, by experiments. And in order to ſatisfy myſelf of the accuracy of thoſe made by others, I have taken the trouble of making a conſiderable number: a few of which I will inſert.

From the experiments made by ſome authors, it appears, that the velocity of effluent water is much leſs than aſſigned by the Newtonian theory: ſo much leſs, that they have concluded the velocity ought to be the ſame as that acquired by a body falling through a ſpace equal to half the depth. But this muſt be erroneous; as the obſerved, or real velocity, can never exceed that aſſigned by a juſt theory, whereas it does exceed that aſſigned by this theory, but from impediments, may be expected to fall ſhort, more or leſs. When I began to make experiments, I wiſhed, as much as poſſible, to get clear of theſe obſtructions, &c.; and, from the recommendation of authors, I began with making the apertures in plates about $\frac{1}{40}$th of an inch thick: the reſult of which is contained in the following Table, where it appears that the obſerved velocity, at the depth of 8 feet, falls $4\frac{1}{2}$ feet ſhort of the computed.—I afterwards tried various forms, all of which gave more water than the thin plate. But as my view was to obtain the greateſt velocity, I ſhall take no notice of any but the cone, which I found to be the beſt. This I made of braſs, about $\frac{1}{10}$th of a foot in length, and the top diameter about twice as much as the bottom, or aperture. The ſtream diſcharged through this is remarkably different from that diſcharged through the plate. This appears like a piece of cryſtal glaſs, to the diſtance of ſome feet, more or leſs, according to the head; neither has it that contraction juſt below the aperture as obſerved in one flowing through a thin plate; and the obſerved velocity at the depth of 7 feet, only falls $1\frac{1}{4}$ foot ſhort of the computed.

PROCESS.

The aperture was firſt bored near the intended ſize; it was then finiſhed with a piece of hard ſteel wire, one end of which was made like a ſquare pyramid, and the other fixed in the ſpindle of a lathe. The wire was afterwards broken into a number of pieces, which were placed cloſe to each other on a diagonal ſcale, and the ſum of their diameters obſerved; they were alſo meaſured by a ſcrew micrometer: all the different pieces were meaſured by ſeveral gentlemen, who never differed ſo much as $\frac{1}{1000}$th part of an inch in the diameter of a piece. And from the velocity of the water through the different holes, or thoſe made by different wires, I have reaſon to conclude that their diameters were all meaſured to the greateſt degree of accuracy.

To obtain the quantity diſcharged in a given time.—An aſſiſtant cloſed the aperture with a finger, till the tube was full, and held the pendulum of a half-ſecond time-piece with the other hand; then, removing both hands at the ſame inſtant, the water was ſuffered to flow as long as convenient, (by a conſtant ſupply, the head of water was always the ſame, during the experiment) and was ſtopped at the end of any given time, to the greateſt degree of accuracy.

To find the velocity.—The water diſcharged was weighed, reduced into cubic inches, and divided by the area of the aperture, in inches; by which the length of the ſtream run out in the whole time, was known.

Experiments, with the aperture made in a plate about $\frac{1}{40}$th of an inch thick.

TABLE I.

Number.	Head, in feet.	Area of the hole, in inches.	Weight run out in a minute, in ounces avoirdupoiſe.	Velocity per ſecond, by computation.	Velocity per ſecond, by experiment.	Difference, at the depth of 16 feet.
1	1.375	.0055	16.97	9.38	7.4	6.7
2	2.2	.0055	21.4	11.86	9.3	
3	4.5	.0055	30.6	16.9	13.35	
4	8.	.0055	40.8	22.62	17.8	
5	1.375	.01227	37.94	9.38	7.42	6.8
6	2.2	.01227	48.1	11.86	9.41	
7	4.5	.01227	68.4	16.9	13.37	
8	8.	.01227	92.2	22.62	18.1	
9	3.66	.0046	23.17	15.31	12.08	6.6
10	8.	.0046	34.39	22.62	18.	
11	3.66	.007	35.36	15.31	12.12	6.5
12	8.	.007	90.9	22.62	18.03	
13	3.66	.09	454.	15.31	12.106	6.6
14	8..	.09	674.	22.62	17.97	

In the ſix following experiments the water paſſed through a cone, $\frac{1}{10}$th of a foot in length.

TABLE I. continued.

Number.	Head, in feet.	Area of the hole, in inches.	Weight run out in a minute, in ounces avoirdupoiſe.	Velocity per ſecond, by computation.	Velocity per ſecond, by experiment.	Difference, at the depth of 16 feet.
15	3.61	.0046	26.87	15.2	13.9	2
16	3.61	.007	41.16	——	14.23	
17	3.61	.0141	85.3	——	14.52	
18	7.	.0046	37.43	21.16	19.53	2
19	7.	.007	57.4	——	19.68	
20	7.	.0141	117.	——	19.91	

The following experiments, were made by the Abbé Boſſuet; his reſervoir was 3 feet ſquare within; the orifices through which the water iſſued, were made in braſs plates, about $\frac{1}{24}$th of an inch thick.

N. B. The meaſures are taken by the Paris foot.

Height of the water above the center of the hole, 11 *feet* 8 *inches and* 10 *lines.*

TABLE II.

By a circular orifice,	6 lines in diameter,	2311	cubic inches
	1 inch in diameter	9281	[per min.
	2 inches do.	37203	

By

By a ſquare orifice,	1 inch by 3 lines	2933	cubic inches
	1 inch by 1 inch	11817	[per min.
	2 inches ſquare	47361	

Height of the water, 9 feet.

By a circular orifice,	6 lines in diameter,	2018	cubic inches
	1 inch do.	8135	[per min.

Height of the water, 4 feet.

By a circular orifice,	6 lines in diameter,	1353	cubic inches
	1 inch do.	5436	[per min.

In Engliſh meaſure, the head to the ſix firſt experiments is 12.5104 feet; the computed velocity 28.296 feet; and the obſerved, at a mean, is near 17½ feet.

Experiments from F. D. Michelotti's *Sperimenti Idraulici*, 1767. The reſervoir was 20 feet high, and 3 feet ſquare within, and had openings at different diſtances below the top. It was ſupplied by a canal 2 feet wide, the bottom of which was horizontal. The water ran into a ciſtern, whoſe area was 289 ſquare feet, of an uniform figure, and the quantity was meaſured by taking its height in this ciſtern.—The meaſures are French.

TABLE

TABLE III.

Size and kind of hole.	Head above the center of the hole.				Time of running.	Cubic feet of water expended.		
	Feet.	Inch.	Lin.	Pa.	Minutes.	Feet.	Inch.	Lin.
Square, of 3 inches.	6	7	4	3	10	463	7	3
	6	10	2	8	12	566	5	6
	11	8	1	6	8½	516	9	5
	11	9	9	10	10	612	1	5
	21	8	3	6	5	415	5	3
	21	8	7	0	6	499	2	8
Square, of 2 inches.	6	7	6	0	15	329	9	8
	11	5	1	4	15	423	5	7
	21	5	3	7	10	385	4	0
Square, of 1 inch.	6	9	1	0	30	158	6	7
	11	10	8	1	24	163	9	6
	21	6	1	0	60	562	11	4
Circular, of 3 inches diameter.	6	8	4	0	15	542	10	6
	11	7	1	0	12	570	11	8
	21	7	4	0	8	521	3	7
Circular, of 2 inches diameter.	6	9	5	0	30	488	8	3
	11	8	8	0	28	589	6	5
	21	10	10	0	20	575	5	10
Circular, of 1 inch diameter.	6	10	6	0	60	247	4	3
	11	8	11	0	60	324	1	5
	22	0	2	0	60	444	6	5

In the firſt experiment, the head, reduced to Engliſh meaſure, is 7.049 feet, and the computed velocity 21.2 feet. The quantity run out in 10 minutes, in French meaſure, is 463.604 cubic feet; the area of the hole is .0625 feet, by which divide the quantity, and we have 7416.06, the feet run out in 10 minutes, which is equal to 12.36 French, or 13.175 Engliſh feet, in one ſecond; or, the true quantity is to the computed, nearly as 5 to 8: and nearly the ſame reſult is drawn from all the experiments made by this author, of which I have copied but a few.

Experiments by Meſſrs. Brindley and Smeaton; the water extended over a large ſurface, the depths meaſured from that ſurface to the top of the holes.

TABLE IV.

Size of the hole.	Head, in feet.	Time in which 20 cubic feet run out.
1 inch ſquare.	1	9 min. 22 ſec.
	2	6 40
	3	5 20
	4	4 44
	5	4 14
$\frac{1}{2}$ inch ſquare.	6	17 33

TABLE V.

Notches 6 inches wide.	Time in which 20 cubic feet run out.
1 inch deep	7 min. 16 ſec.
$1\frac{3}{8}$	4 55
$2\frac{5}{16}$	2 19
$3\frac{1}{8}$	1 33
$6\frac{1}{2}$	0 30
5	0 46
$1\frac{1}{4}$	5 26
$1\frac{5}{8}$	3 55
$5\frac{3}{8}$	0 42

Experiments

Experiments by Profeſſor Helſham, of the univerſity of Dublin.

TABLE VI.

Head, in feet.	Diameter of the hole.	Time, in ſeconds.	Weight, in grains.
4	$\frac{1}{10}$ inch	10	2944
4	$\frac{4}{10}$	—	47040
4	$\frac{5}{10}$	—	72960
4	$\frac{8}{10}$	—	178560
2	$\frac{1}{10}$	—	2087
2	$\frac{4}{10}$	—	33600
2	$\frac{5}{10}$	—	51840
2	$\frac{8}{10}$	—	128400

In the firſt fourteen experiments in Table I. the velocity, or quantity diſcharged, is ſomething greater than in the following tables, though the orifices were made in flat plates; this is owing to the form of the lower part of the tube being the fruſtum of a cone, and the plate being half an inch in diameter. In the ſix laſt experiments, where a cone is ſubſtituted for the plate, we have the velocity conſiderably increaſed: ſo much as, at the depth of 16 feet, to fall only 2 feet ſhort of the computed velocity. But if the laws of hydroſtatics aſſigned no greater velocity to flowing water, than that which would be acquired by a heavy body in falling through half the depth, we ſhould have an effect produced without a cauſe. For the velocity at the depth of 16 feet ſhould be, according to that law, the ſame as a body would acquire in falling through 8 feet, or 22.627 feet per ſecond, whereas it is, in fact, 30 feet. And whatever advantages I may have taken, in reſpect to the form of the containing veſſel, this alters not the preſſure of the fluid, or the laws of nature, but tends to remove the impediments.

But from theſe experiments, we are not to conclude, that the velocity of water, as flowing through orifices made in large ſurfaces, is in the ſame ratio. Sir Iſaac Newton concluded, that the obſerved, or real velocity, was leſs than the computed, in the ratio of 1 to the ſquare root of 2, or as 1 to 1.414; the Abbé Boſſuet, as 100 to 150; and F. D. Michelotti, as 5 to 8, &c.—The quantities in the foregoing tables, reduced to the ſame denomination, ſtand as below:

Newton - - - - -	.707
Boſſuet - - - - - -	.615
Banks - - - - - - -	.750
Michelotti - - - -	.625
Helſham - - - - - -	.705
Smeaton - - - - -	.631
6)	4.033
Mean =	.672

Boſſuet and Michelotti have made experiments upon the largeſt ſcale of any I have heard of, though they do not give the greateſt velocity; yet they were both agreed with Sir Iſaac in the method of computing it: hence, they could not be biaſſed to favour a different theory.—If we take the mean velocity of the above experiments, we find it to be $\frac{672}{1000}$ of the computed; or their ratio is as $5\frac{3}{8}$ to 8: viz. at the depth of a foot, the actual velocity from them is $5\frac{3}{8}$ feet, and in the ſame proportion for any other depth.

EXAMPLE.

Given the depth 6 feet, to find the velocity?

Let d = the depth, in feet.
v = the velocity required.

Then, $.672 \times 8 \sqrt{d} = v$; $\sqrt{d} = 2.449$, which $\times\ 5\frac{3}{8} = 13.163$, the required velocity.

When water is diſcharged through perpendicular ſections, the velocity at the bottom will be ſomething greater than at the top of the orifice, but if the depth or breadth of the orifice be but little, compared with its depth below the ſurface of the dam, we may take the depth of the center of the hole for the mean depth, without ſenſible error.

RULE.

Meaſure the depth in feet, extract the ſquare root of that depth, and multiply it by 5.4, *which gives*

gives the velocity in feet per ſecond; this multiplied by the area of the orifice in feet, gives the number of cubic feet which flows out in one ſecond.

EXAMPLE.

Let the depth be 10 feet, the breadth of the orifice 7 inches, and its horizontal length 4 feet; required the quantity of water expended in one ſecond?

The ſquare root of 10 is 3.162, which multiplied by 5.4, gives 17.0748 feet, the velocity per ſecond. The area of the orifice is $\frac{4 \times 7}{12}$ = 2⅓ feet, by which multiply the velocity, and we have 39.84 cubic feet per ſecond.

If the breadth of the orifice is great, compared with the head, we muſt compute the velocity at the top and bottom, and find the area of a perpendicular ſection diſcharged in a given time.

EXAMPLE.

Given the perpendicular depth of the orifice 2 feet, its horizontal length 4 feet, and its top 1 foot below the ſurface of the water in the dam, to find the quantity diſcharged in one ſecond?

See Fig. XX, where B is the ſurface of the water, BA the depth of the upper ſide of the orifice, and BD the depth of the bottom thereof. BD is the axis of a parabola, AC and DG ordinates which repreſent the velocity of flowing water at theſe diſtances from the vertex; we have, therefore, to find the area of the ſection ACGD, which will be done by finding the area of BDG, and deducting that of BAC from it.

Put DB $= d$, AC $= a$, DG $= n$, BA $= c$. Then the area of BGD $= \frac{2dn}{3}$, and that of BAC $= \frac{2ca}{3}$; hence $\frac{2dn}{3} - \frac{2ca}{3} =$ area of ACGD. For, by the above rule, the velocity at the top is 5.4, at the bottom 9.24, and the area required = 14.82, which multiplied by the length, gives 59.3 cubic feet per ſecond.

Of the quantity of water diſcharged through ſlits or notches cut in the ſide of a veſſel or dam, and open at the top. See Fig. XXI.

If the ſurface is conſidered without motion, and the velocity below increaſing with the ſquare root of the depth, then the area of the perpendicular ſection will be a part of a parabola, and will be found by multiplying the velocity at the bottom by the depth, and taking two-thirds of the product for the area, which again multiplied by the breadth of the notch, gives the quantity or number of cubic feet diſcharged in the given time.

For Example,

Let the depth of the notch be 5 inches, and the breadth 6 inches; required the quantity run out in 46 ſeconds?

The depth in feet is .4166, its ſquare root is .6455, which multiplied by 5.4 × ⅔ = 3.6, gives 2.3238, this multiplied by .4166, the depth, produces 96825, which being multiplied by the breadth, half a foot, or .5, gives .48412 feet per ſecond; this multiplied by the time, 46 ſeconds, gives 22.269 cubic feet.—The ſixth experiment in Table v. gives 20 cubic feet.

For another Example,

Let the depth be 3⅛ inches, the breadth 6 inches; required the quantity which flows over in 93 ſeconds?

Let d = the depth in feet.
b = the breadth in feet.
n = 3.6.

Then, $3.6 db\sqrt{d}$ = cubic feet per ſecond: in this Example, $d = .2604$, $b = .5$, $\sqrt{d} = .51$; theſe multiplied together, produce 22.227 cubic feet: more than the experiment by 2¼ feet, nearly.—At the depth of 1 inch, the computed quantity is 18.727 cubic feet, but by the experiment it is 20. Alſo, through a notch 1 inch ſquare, the quantity, as near as can be meaſured by experiment, is 8.6 cubic feet, and by computation only 7.3. From which compariſons it appears, that all the experiments have not been equally accurate. In the above method of computing, the ſurface of the water is ſuppoſed to be without motion; this is not ſtrictly true in any caſe, but more eſpecially in rivers: though the deviation from experiment, in ſtill water, is not great. And if we wiſh to take the velocity of the ſurface into the computation, we may proceed as in finding the quantity which flows through an orifice juſt below the ſurface.

Let $v =$ the velocity of the ſurface. Then will $\left.\overline{\frac{v}{5.4}}\right|^{2} = h$, the head which would produce that velocity, if this is added to the depth of the notch, we ſhall have the following Theorem, viz. $\frac{2}{3} \times \overline{d+h} \times 5.4\sqrt{d} - \frac{2hv}{3} =$ the area of the ſection, which multiplied by the breadth of the ſection, gives the number of cubic feet per ſecond.

EXAMPLE.

Let $d = 1$ foot, the depth of the notch; $b = 1$ foot, the breadth; $v = 1$ foot per ſecond; $h = \left.\overline{\frac{v}{5.4}}\right|^{2} = .034$.

Then $\overline{d + b} = 1.034$, $\sqrt{d} = 1$, and $\frac{2bv}{3} =$ $.0228$; or $1.034 \times 5.4 \times \frac{2}{3} - 1 \times \frac{2}{3} = 3.7003$, the area; or $\overline{d + b} \times 3.6\sqrt{d} - \frac{2bv}{3} =$ the area which multiplied by b, the breadth, gives 3.7 cubic feet per ſecond.

In this Example, by taking the velocity of the ſurface to be one foot per ſecond, the quantity run out in a ſecond is one-tenth of a cubic foot more than it would have been had the ſurface been ſuppoſed without motion.

On the impulſive force of effluent water.

The preſſure of water, upon a given ſurface, is directly as the depth, whatever may be the figure or magnitude of the containing veſſel or reſervoir: but the velocity is as the ſquare root of the depth, and the impulſe as the ſquare of the velocity, or as the depth. In addition to the demonſtration already given, I ſhall add the following obſervations;

1. The depth of the water may be conſidered as the power which projects or impels the flowing ſtream.

2. The quantity and velocity conjointly, are the effect of that power.

3. Could we conſider the velocity alone, and ſuppoſe the power to act upon a ſingle particle, in that caſe, twice the depth would produce a double velocity in the ſaid particle.

4. Alſo, if a given power communicates a certain velocity to one particle, double that power would communicate the ſame velocity to two particles.

5. Therefore, if a given power gives a certain velocity to one particle, it will require four times that power to communicate two degrees of velocity to two particles. And in water flowing through a given hole, the quantity will increaſe directly with the velocity, viz. a double velocity muſt give a double quantity, and of conſequence the power which produces it is always as the ſquare thereof. From hence we infer, that the effect of a ſtream of water flowing through a given orifice, ought to be directly as the depth of that orifice: that is, if a ſtream of water flows through a hole at the depth of one foot, and produces a given effect; a ſtream flowing through the ſame hole, at the depth of four feet, ought to produce four times the effect, though the expence of water be but double.

The abſolute force of a ſtream has commonly been eſtimated by the weight of a column or priſm of water, the ſection of which is equal to the area of the orifice: this is the true meaſure of the preſſure againſt a given ſurface, but the water in motion ought to produce a greater effect.

In order to compare the impulſe of a given ſtream with the weight of the column which impels it, I made the experiments in the following Table.

FIG. XXII.

The lever or ſteel-yard NL, moveable on its center C, was ſo fixed, that a point in the plate N, (the head of a hook *b* riveted into it) was exactly under the ſtream *s*, and the moveable weight *w* ſhifted till the force of the ſtream would raiſe the end L from the prop P. When the ſtream was ſtopped, or the lever removed, weight was applied to the hook *b*, till the end L was raiſed from the prop, as by the ſtream; by which means the force of the ſtream, compared with a dead weight, was obtained.—In the third column of the Table, we have the weight of a cylinder of water, the length of which is equal to the depth in the firſt column, and the ſection equal the area of the hole in the ſecond; and while the hole remains the ſame, the weight muſt be exactly as the depth; and alſo in the fourth column, the force, though more than the weight, is ſtill directly as the depth; or, the forces, as meaſured by experiments, though they differ a little, are near enough to prove that they increaſe with the ſquare of the velocity, or with the depth, directly.

TABLE.

Depth, in feet.	Area of the hole.	Weight of the perpendicular column.		Force on the ſteel-yard.	
		dwts.	grs.	dwts.	grs.
1.5	.0055	0	25	2	0
2.0833	.0055	1	11	2	18
8.	.0055	5	13½	10	10
1.25	.02259	3	14	6	0
1.5416	.02259	4	10	7	6
8.	.02259	22	21	38	6
2.375	.0141	4	6	7	0
3.458	.0141	6	4	10	6
7.0416	.0141	12	13½	21	10
9.3125	.0141	16	14.8	28	5

But the force of a ſtream, or rather the impulſe, cannot be meaſured, by its effect on a wheel, when that wheel is confined in a channel; for in that caſe, the ſtream, when it has ſtruck the float-boards, cannot make its eſcape, and is therefore heaped or raiſed againſt the floats much higher than it would be if at liberty to paſs.

On emptying of veſſels.

The velocity with which water flows through orifices, may be obtained by knowing the time in which a cylinder, &c is emptied by the water, running out through a hole in the bottom.

Let $x =$ the depth, in feet.
$z =$ the area of the falling ſurface, in inches.
$n =$ the area of the ſtream, or orifice, in inches.
$s =$ 16 feet, the ſpace through which a body deſcends, by gravity, in one ſecond.
$m =$ 32 feet, the velocity acquired at the end of one ſecond.

Then, As $\sqrt{s} : m :: \sqrt{x} : \frac{m\sqrt{x}}{\sqrt{s}}$, the velocity of the ſpout, per ſecond, at the firſt inſtant. And, As $z : n :: \frac{m\sqrt{x}}{\sqrt{s}} : \frac{nm\sqrt{x}}{z\sqrt{s}}$, the velocity of the falling ſurface per ſecond. Alſo, As $\frac{nm\sqrt{x}}{z\sqrt{s}} : 1 :: \dot{x} : \frac{z\sqrt{s}\,\dot{x}}{nm\sqrt{x}}$, the fluxion of the time of emptying; the fluent of which is $\frac{2z}{nm}\sqrt{sx}, = \frac{z\sqrt{x}}{n\sqrt{s}}$.

Experiment 1. A cylindrical tube 3.166 feet long, and 2.705 inches diameter, was exhauſted in 3 minutes and 16 ſeconds, through a hole the area of which was .0141.

Experiment 2. The area of the cylinder and orifice the ſame, the length of the tube 6.458 feet, the time of exhauſting, by obſervation, is 4 minutes and 40 ſeconds.

In Experiment 1, $\frac{a\sqrt{x}}{n\sqrt{s}} = \frac{10.23144}{.0564} = 3$ minutes and 1½ ſecond, which would be the true time, if at the depth of 16 feet the velocity was 32 feet per ſecond. In Experiment 2, $\frac{a\sqrt{x}}{n\sqrt{s}} = \frac{14.605668}{.0564} = 4$ minutes and 19 ſeconds, the time of exhauſting, by theory.—From theſe experiments, to find the velocity per ſecond, when $a = 21$. If $t =$ the time of exhauſting, we have, from the above expreſſion, $\frac{a\sqrt{x}}{n\sqrt{s}} = t$; from which we find $s = \left(\frac{a\sqrt{x}}{tn}\right)^2$, and from the firſt experiment is equal to $\left(\frac{10.23144}{2.7636}\right)^2 = 13.6974$ feet: in the ſecond experiment, $s = \left(\frac{14.605668}{3.948}\right)^2 = 13.6863$ feet.

In theſe experiments there is no difficulty in obſerving the time to half a ſecond, or nearer. The length of the tubes can alſo be truly meaſured; but to obtain the area to a greater degree of accuracy, the water was weighed, reduced into inches, and the ſection computed. Taking the mean of theſe two experiments, at the depth of 13.6918 feet, the velocity is 27.3836 feet per ſecond. And As $\sqrt{13.6918} : 27.3836 :: \sqrt{16} : 29.6$ feet, the velocity at 16 feet, according to theſe experiments.

Experiments

Experiments on the velocity with which water flows through ſyphons.

FIG. XXIII.

AB is a veſſel, &c. containing water, filled to the line *ss*; *svb* a bended tube, or ſyphon, *sv* the ſhorter, *vb* the longer leg; *vs* the perpendicular altitude of the crown of the ſyphon above the level of the water to be conveyed through it; *sb* the perpendicular height of the difference of the two legs: the whole being filled, this difference is the power which gives motion to the whole; for if both ends were placed in the ſame veſſel of water, whatever might be their lengths, their endeavour to deſcend would be as the perpendicular heights, and theſe being equal, there could be no motion. When *v* is not elevated more than 30 feet, the water in *vb* begins to deſcend, and the weight of the atmoſphere cauſes that in the veſſel AB to follow up the tube *sv*.

Perpendicular height of the ſyphon above the level of water.	Perpendicular length below the level.	Area of the orifice.	Weight per minute in ounces avoirdupois.	Computed quantity per minute.	Square root of *sb*.
21 In.	84.	.0141	3.75	3.88	4.899
39	87.5	—	4.075	4.158	5.244
58	26.5	—	4.095	4.064	5.147
62	55.75	—	5.819	5.904	7.466
62	35.75	—	4.63	4.76	5.98
97	9.25	—	3.245	2.41	3.041
95½	20.	—	3.117	3.542	4.472

Without regard to the height of the top of the ſyphon above the water, the quantity or weight diſcharged is nearly as the ſquare root of the height *sb* For, though the weight of the water in the tube does for a moment retard the velocity, yet when once put in motion, that reſiſtance, by theſe experiments, is not worth notice. And while the top keeps free from air, we may compute the velocity by taking the difference of the legs, and proceeding in the ſame manner as in finding the velocity through the bottom or ſide of a veſſel.

Experiments on the velocity with which water aſcends into a vacuum, by the preſſure of the atmoſphere.

To make theſe experiments, I procured a number of tubes, of different lengths, the whole when ſcrewed together was near 40 feet long; theſe were fixed in a ſtair-caſe, in ſuch a manner that I could make uſe of the whole or of a part, as the experiment required; the higheſt was furniſhed with a ſtop cock at the top, ſo that when the water was raiſed by a ſmall pump, by turning the cock, the tube below always remained full of water. Next, the receiver, Fig. XXV. was exhauſted, accurately weighed, then ſcrewed upon the pipe or tube, the cock opened, and the water ſuffered to flow into the vacuum for any convenient time, which was meaſured by a time-piece. The receiver was then unſcrewed, and weighed again, by which the weight of the water which entered in a given time was known; and as the area of the orifice was alſo known, the velocity was eaſily computed.

Number.	Height at which the water [illegible] found.	Time, in ſeconds.	[illegible]	Cubic inches.	[illegible]	Velocity per [illegible]
1	8 inches	48	26	44.92	.002356	36.95
2	8	45	27.5	47.52	——	37.85
3	8	60	36	62.2	——	36.66
4	10	36	22	38	——	37.33
5	10	44	27	46.65	——	37.43
6	4.5 feet	36	20	34.56	——	33.95
7	4.5	47	27	46.65	——	34.3
8	7.4	50	27	46.65	——	33.06
9	7.2	49	26.4	45.62	——	32.9
10	7.2	51	27.2	47	——	32.71
11	7.5	60	32	55.29	——	32.59
12	13.8	60	26.3	45.44	——	26.78
13	13.8	60	26.5	45.79	——	26.99
14	18.2	60	23.7	41	——	24.17
15	25	60	16.5	28.51	——	16.8
16	25	60	17	27.64	——	16.5
17	32	60	6.76	11.68	——	6.89

When theſe experiments were made, the barometer ſtood from 29.5 to 30 inches, and as the preſſure is frequently varying, the velocity of the riſing water muſt alſo vary. At the ſurface of the water, under the greateſt and leaſt preſſure,

ſure, the velocities would be in the ratio of 37 to 36. At the altitude of 16 feet, as 36 to 2 At the height of 26 feet, as 23 to 17, &c. In theſe experiments, the area of the hole is ſmall, compared with the area of the lower part of the tube: if the tube was equally wide, as in the lower part of a pump, the velocity of the water would be much leſs. And when a column of water is to be moved by the weight or ſpring of the air, as in a pump, the water is not inſtantaneouſly put in motion; and the longer the tube, the longer is the water of acquiring its greateſt velocity.

N. B. In the above experiments, I frequently made the vacuum without an air-pump. Upon the tall tube, above the cock, I ſcrewed as much more tube, as made the whole height about 3 feet; then an open receiver, furniſhed with a ſtop cock at the top, was ſcrewed upon the tube, the whole was filled with water, and covered with leather and a plate, with the jet-pipe and cock; after this, the cock in the tube was opened, and the water deſcended out of the receiver and down the pipe or tube, till the preſſure of the air could ſupport it: the cocks being turned, the receiver was unſcrewed, perfectly exhauſted.

A TREATISE

A TREATISE ON MILLS.

PART IV.

EXPERIMENTS ON CIRCULAR MOTION.

In this part, many of the principal theorems and problems contained in the firſt and ſecond parts of this work are confirmed by experiments.

The machine, Fig. XXIV, is uſeful for making experiments on circular motion; and alſo for comparing the effects produced by a given weight in falling through a given ſpace, when applied at different diſtances from the axis, &c. The cylinder A is 2 inches and B 4 inches diameter. The wheel W has 84 teeth, and the pinion P has 8. The balls c c may be of any weight, and fixed at any diſtance from the center G. In a rod or arm revolving round a center, it will be, as the velocity of any point in that arm is to its diſtance from the center, ſo is the velocity of any other point to its diſtance, &c. The balls C C make 10½ turns, while the cylinder A makes 1; therefore, if the diſtance at which they are placed be multiplied by 10½, the product will be the diſtance at which a body muſt be placed from the axis of the cylinder A; and to revolve with it, the velocity of which ſhall be the ſame as that of the ball C.

For Example,

Let the balls be at the diſtance of 6 inches from G, this multiplied by 10½, gives 63, at which diſtance a body revolving round with the cylinder A, would have the ſame velocity with the balls C C. But their central forces, being as the ſquares of the velocities divided by their diſtances, would be different: in this caſe, as $\frac{1}{6} : \frac{1}{63}$, for their velocities are equal, or $V = v$, but their forces are $\frac{V^2}{D}$ and $\frac{v^2}{d}$.

In the machine, the wire arms are each 8 inches long, and weigh 22 dwts. together; the diſtance of the center of gyration is 4.61 inches from their center, where the reſiſtance is equal to their weight, but reduced to the end of the arm is only equal to 7.33 dwts.; and at any diſtance d the reſiſtance will be $= \frac{\overline{4.61}^2 \times 22}{d^2}$.

EXAMPLE.

Let $r =$ the radius of the cylinder from which the weight M is ſuſpended.
$s =$ the diſtance GC × 10.5.

Then will $\frac{Mr^2}{s^2}$ be the force exerted at C to give motion to the balls.

Let $r = 1$.
$s = 84$.
$M = 8$ oz.
$C + C = 8$ oz.

Then will $\frac{Mr^2}{s^2} = .0011336$ oz. the moving power; the reſiſtance is, the moving power itſelf, ($= .0011336$) the arms reduced to the end, ($= .3666$) and the balls, ($= 8$), the ſum is 8.3677996 oz., the whole reſiſtance. If the moving power is divided by this, we have $\frac{.0011336}{8.3677996} = .0001356$, the accelerating force compared with gravity, which multiplied by 16, gives .0021469 feet the firſt ſecond. And As $1''^2 : .0021469 :: 100^2 : 21.469$ feet, the ſpace through which the body would deſcend in 100 ſeconds. By which proceſs the following experiments may be compared with each other, and with the theory.

For, let w = the weight of the fly.
t = the time of deſcent, in ſeconds.
F = feet paſſed over by M in the time t.
d = 16 feet.
n = the ſpace paſſed over by the fly, in the time t.

Then will $n = \frac{Msr}{ws^2 + Mr^2} \times dt^2$, Theo. 1.

$F = \frac{Mr^2}{ws^2 + Mr^2} \times dt^2$, Theo. 2.

$$\left.\begin{aligned} w &= \frac{Mdr^2t^2}{Fs^2} - \frac{Mr^2}{s^2}, \\ \text{or } w &= \frac{Mrdt^2}{ns} - \frac{Mr^2}{s^2}, \end{aligned}\right\} \text{Theo. 3.}$$

$M = \frac{Fws^2}{dr^2t^2 - Fr^2}$, Theo. 4.

$t = \sqrt{\frac{Fws^2}{Mdr^2} + \frac{F}{d}}$, Theo. 5.—See Prob. XV, Part II.

In theſe theorems, the reſiſtance of the air is not conſidered; but the following experiments will be ſufficient to convince us that when the moving body is ſmall, and the velocity great, it will make a difference in the concluſion, as it prolongs the time, &c. And although it is of little importance to conſider its effects on heavy bodies which move ſlowly, yet in making experiments to prove the truth of a theory, it becomes neceſſary to take notice of every impediment. The weights ſcrewed upon the arms were 2.5 inches long, 1 broad, and a little thicker at the middle than at the ends (ſee Fig. XXVI.); theſe could be fixed ſo that either the end or the flat ſide ſhould ſtrike directly againſt the air.

Experiment 1. The fly being fixed to meet with the leaſt reſiſtance, a weight of 5.7 oz. ſuſpended from the cylinder A, deſcends through 9 feet in 62 ſeconds.

Experiment 2. The fly being placed in a vertical poſition, ſo that the flat ſide might preſs againſt the air, 15 oz. ſuſpended from the cylinder A, deſcends through 9 feet in 62 ſeconds; but when the falling weight has acquired a velocity of 2 inches per ſecond, it ceaſes to be accelerated, or moves with an uniform motion.

Experiment 3. The fly being placed or fixed to meet with the leaſt reſiſtance, 7.3 oz. at the cylinder A deſcends through 5 feet in 40 ſeconds, at the cylinder B through 5 feet in 20 ſeconds.

Experiment 4. The flat ſide being ſet to move againſt the air, 7.3 oz. ſuſpended from B falls through 5 feet in 23.5 ſeconds, but ſuſpended from A, in 55 ſeconds.

Obſervations.—1. When the velocity is the ſame, the reſiſtance, ſo far as it ariſes from the friction and inertia of the machine, muſt be the ſame. 2. In Experiments 1ſt and 2d, the velocities are the ſame, but the weights or powers which produce them are 5.7 and 15; hence their difference 9.3 is employed to overcome the reſiſtance of the air. 3. In both experiments, when the motion is uniform, the velocity of the fly is 14 feet per ſecond; and 9.3 oz. divided by 84, gives .11071 oz., the force exerted on the fly to overcome the reſiſtance of the air. 4. The difference of ſurface oppoſed to the air in the two experiments is 4.7 inches, by which divide .11071 oz. and we have .0235 oz., the reſiſtance againſt one inch; by computation it is .02123 oz. 5. The reſiſtance ariſing from the air being as the ſquare of the velocity, the 3d and 4th experiments will be found to agree with each other, and with theory, but will not if the reſiſtance of the air is neglected.

In the following experiments, the fly is ſo fixed as to meet with the leaſt poſſible reſiſtance.

TABLE I.

No.	Power. M.	Cylinder. r.	Fall, inches. F.	Time, ſec. t.	Turns of the fly.
5	7.3	A	90	52	152
6	7.3	B	90	26	76
7	7.3	B	22½	13	19
8	7.3	A	22½	26	38
9	14.6	A	90	36½	152
10	14.6	A	45	26	76
11	29.2	A	90	26	152

The reſiſtance of the machine is found by Theo. 3. If the fly is uſed, its weight muſt be ſubtracted from *w*, and a weight is left which is equal to the whole reſiſtance reduced to the place of *w*. But in theſe experiments, the weight and magnitude of the fly remain the ſame.

1. It appears (No. 5 and 6), that by increaſing the diſtance at which the power is applied, we diminiſh the time of deſcent, but the power in falling through a given ſpace, produces effects which are as the times of deſcent.

2. When different powers are applied to the ſame cylinder, as in the 5th, 9th, and 11th experiments, they produce effects which are directly as the ſquare roots of theſe powers, and inverſely as the times. For, As $\sqrt{7.3} : \sqrt{14.6} :: 36\frac{1}{2} : 52$, nearly; and As $\sqrt{7.3} : \sqrt{29.2} :: 26 : 52$. The effects are the ſame, and the times are inverſely as the ſquare roots of the

powers, or the powers are inverſely as the ſquares of the times, or directly as the ſquares of the velocities.

3. From the 5th and 10th it appears, that if a given weight in falling through a given ſpace produces a certain effect, double that weight in falling through half the ſpace produces half the effect.

4. In the 11th experiment the velocity is twice as great as in the 5th, but 4 times the power is employed to produce it: that is, the power is as the ſquare of the velocity.

5. When equal effects are produced in the ſame time, the powers will be inverſely as the diſtances at which they are applied, but the ſpaces deſcended through will be directly as the ſaid diſtances. See experiments 6th and 10th.

TABLE II.

No.	Power, in ounces.	Cylinder.	Weight of the fly, in ounces.	Radius of the fly, in inches.	Fall, in feet.	Time, in ſeconds.	Force acting on the fly, in ounces.	Turns of the fly.	Space paſſed over by the fly.	Weight of the fly, multiplied by the ſpace.	Relative velocity of the fly.	Centrifugal force of the fly.
12	4	B	4	8	6	26	.0952	61	254	1016	2	2
13	4	B	16	4	6	26	.1904	61	127	2032	1	4
14	4	B	8	5.65	6	26	.1346	61	179.8	1438.7	1.414	2.8
15	16	A	4	8	6	26	.1904	122	508	2032	4	8
16	4	A	4	4	6	26	.0952	122	254	1016	2	4
17	1	B	4	4	6	26	.0476	61	127	508	1	1
18	2	A	4	4	3	26	.0476	61	127	254	1	1

In making theſe experiments, when the cylinder B was uſed, there was an addition of 11 dwts., and when A was uſed of 44 dwts., to the power; which weights alone would turn the machine, and deſcend through 6 feet in 26 ſeconds: ſo that the power in the table was wholly exerted in turning the fly.

The numbers under *Force acting on the fly*, are obtained by multiplying the radius of the cylinder uſed by the power, and dividing by the radius of the fly multiplied 10.5. For example, No. 12; the radius of B is 2 inches, which multiplied by the power, 4 oz., gives 8, which divided by 84 (10.5 × 8), the radius of the fly, the quotient is .0952, and ſo for the reſt.

The products of the weight of the fly into the ſpace it has deſcribed, are always as the force acting on the fly: that is, when the power employed is, by the above proceſs, reduced to the fly, the effects are as theſe powers, whether the fly be heavy or light, or whether the circle it deſcribes be large or ſmall.

The ſpace paſſed over by the fly is obtained by multiplying the circumference of the circle in which it moves, by the number of turns.

The relative velocity, is the ſpace reduced to leſs numbers, by diviſion.

The centrifugal forces are computed by the theorems in the firſt part; but here the numbers are not any real meaſure, but comparative; for as the velocity is conſtantly increaſing, ſo is the central force. At the end of the time, the true centrifugal force, in experiment 12th, is nearly 17 times the weight of the revolving body, or 68 oz.; and the true force, in ounces, at the end of the other experiments, will be had by multiplying that in the table by 34.

When the cylinder B is uſed, the angular velocity is always the ſame; as it is alſo when A is uſed (except No. 18), but is twice as great as when B is uſed, or the revolutions are performed in half the time.

When the power, and diſtance at which it is applied, increaſe with the radius of the fly, the angular velocity will remain the ſame, No. 12, and 18; for 3 feet of cord uncoiled from A, will produce as many revolutions as 6 feet from B.

The velocities are as the ſquare roots of the powers which produce them, or the powers are as the ſquares of the velocities, No. 12 and 17.

When any number of revolving bodies, each multiplied by the ſquare of its diſtance from the center, have their products equal, by a given power, equal angular velocities will be produced, No. 12, 13, 14, 17.

If a given power communicates a certain velocity to one quantity of matter, twice, thrice, &c. that power will communicate the ſame velocity to twice, thrice, &c. that quantity of matter, No. 13 and 17.

If the weight of a revolving body, and time of a revolution, remain the ſame, the power will be as the ſquare of the diſtance, No. 12 and 17.

The centrifugal forces are as the impelling powers, when theſe powers are applied in the ſame manner, No. 13 and 17; for the radius of the circle being the ſame, the forces are as the ſquares of the velocities, or as the powers which produce them. See Prob. XIII, XIV, &c., Part II.

Experiments on Water-wheels.

FIG. XXVII.

ABCD is a water-wheel with buckets, the circumference is 5.6 feet, or the diameter of the circle paſſing through the center of the buckets is 20 inches. By turning a ſcrew, this wheel may be taken off the axle, and one of a different ſize put in its place. The cog-wheels may alſo be changed at pleaſure; as alſo the wheels or trundles on the ſecond axle, which are turned by the cog-wheels. On the ſecond axle, is another cog-wheel, with 84 teeth; this turns an upright ſpindle, with 8 teeth or leaves; on the top of this ſpindle is fixed a fly, the weight and radius of which may be varied at pleaſure.

FIG. XXVIII.

The plate FF is divided into 60 parts, and the index I makes one revolution while the water-wheel makes 60. The index L is fixed on the axle of the water-wheel, and therefore makes the ſame number of revolutions: the plate GG, over which it moves, is divided into 10 parts. There is alſo another index, not viſible in the figure, which makes one revolution while the water-wheel makes 600.—By theſe, and a time-piece, to meaſure the minutes and ſeconds, &c., the number of revolutions, made by the water-wheel, in any given time, may eaſily be known.

The water is conveyed from the cock N into the trough H, or into the veſſel FGE, which has three apertures or ſluices at 1, 2, 3; theſe apertures are ſo adjuſted by trial, that when the veſſel is filled to FG, if any one is opened, it will diſcharge the water exactly as faſt as it is ſupplied by the cock; ſo that the water acts upon the wheel both by weight and impulſe. The ciſtern which ſupplies the cock is always kept equally full; that is, the water is returned by a pump, and the perſon who works it takes care to keep the ſurface always at the ſame mark.

ON THE APPLICATION OF WATER.

ON this ſubject there have been various opinions; but opinions unſupported by theory or experiment, claim little regard. At whatever altitude the water might be applied to the wheel, it has been the practice with many to diminiſh that altitude, in order to obtain a head, as it has been ſuppoſed that the impulſe produced by that head would have a greater effect than the water by its gravity could produce, if thrown into the buckets immediately from the ſurface. And no doubt but it is owing to opinions of this kind, that the water, which might be applied at the top, is often cauſed to ſtrike, or fall upon the wheel, about 45 degrees from the vertex.—The following experiments, which have been often repeated, both in public and private, and with the greateſt accuracy and attention, give no encouragement to diminiſh the head for the ſake of obtaining more advantage by the impulſe.

Whether the effect is meaſured by the velocity, its ſquare, or its cube, every thing remaining the ſame, it will be granted, that when the wheel moves quickeſt, the water is applied to the greateſt advantage.

Experiment 1. The orifice, No. 1, is opened, the water ſtrikes the wheel near the bottom, and acts by impulſe and gravity; the head is equal to the diameter of the wheel. The velocity of the ſtream is 6.5 feet per ſecond; the water-wheel makes 60 revolutions in 7 minutes and 21 ſeconds, or is turned 8.2 times in one minute.

Experiment 2. The ſluice No. 2. is opened, the water ſtrikes the wheel near the center, and cauſes it to revolve 60 times in 3 minutes and 57 ſeconds, or 15.19 times in a minute.

Experiment 3. The orifice No. 2. is ſhut and No. 3. opened, the water falls upon the wheel, with impulſe, at 45 degrees from the vertex. The wheel revolves 60 times in 3 minutes and 28.5 ſeconds, or 17.26 in a minute.

Experiment 4. The water is conveyed into the trough H, from which it falls upon the vertex of the wheel; 60 revolutions are made in 3 minutes and 15 ſeconds, or 18.46 in a minute.

In theſe experiments, the fly makes 21 revolutions for 1 of the water-wheel, its weight was 4 oz. and diameter 13 inches. The comparative velocities of the water-wheel or fly, will be as the number of turns made in a given time, or as 8.2, 15.19, 17.26, and 18.46.

In the following table we have the whole in one view.

TABLE III.

Experiments.	Sluice.	Turns of the water-wheel per minute.	Relative velocity.	Turns of the fly per minute.	True velocity of the fly, in feet, per ſecond.	True velocity of the water-wheel, in feet, per ſecond.	
1	No. 1	8.2	1	172.2	11.96	.765	
2	2	15.19	1.85	319	22.15	1.417	
3	3	17.26	2.1	362.5	25.17	1.61	
4	overſhot	18.46	2.25	387.6	26.91	1.723	
5	No. 1	4	1	168	11.66	.373	In theſe four experiments, the fly revolves 42 times for the water wheel once; its radius 8 inches.
6	2	7.69	1.92	323	22.43	.717	
7	3	8.88	2.22	373.3	25.92	.829	
8	overſhot	9.52	2.38	400	27.77	.888	
9	No. 1	3.05	1	128.1	8.89	.284	In theſe, the radius of the fly is 8 inches, weight 12 oz.; revolutions 42 to 1.
10	2	6.38	2.09	268	18.61	.595	
11	3	7.28	2.38	305.7	21.23	.679	
12	overſhot	7.72	2.53	324.2	22.51	.720	
13	No. 1	2.14	1	134.8	9.36	.199	The radius as above; weight 4 oz.; revolutions 63 to 1.
14	2	5	2.33	315	21.87	.466	
15	3	5.84	2.72	368	25.55	.545	
16	overſhot	6.4	3	403.4	28.01	.597	

In the above four ſets of experiments, which are conſiderably varied, a given quantity of water produces the greateſt number of revolutions when applied at the top of the wheel. The ratio of the revolutions when the water is applied at the top, is to the revolutions when applied at the quarter, nearly as 32 to 30, or as 16 to 15. For, in the firſt ſet, As 2.25 : 2.1 :: 16 : 14.9. In the ſecond ſet, As 2.38 : 2.22 :: 16 : 14.9. In the third ſet, As 2.53 : 2.38 :: 16 : 15.05. In the fourth ſet, As 3 : 2.72 :: 16 : 14.5.

From theſe experiments it appears, that for every 15 revolutions when the water is applied at the quarter, there will be 16 when the water is applied at the top of the wheel.

Experiments on a wheel 10 *inches diameter.*—The water applied at the top produces 60 turns in 4 minutes and 25 ſeconds, but when applied to the quarter, 60 turns in 4 minutes and 52 ſeconds; the turns per minute in the firſt caſe are 13.58, in the ſecond 12.32; and As 13.58 : 12.32 :: 16 : 14.51, the number of revolutions, when the water is applied at the quarter, for 16 when it is applied at the top, or a given quantity of water in the overſhot produces 16 turns of the water-wheel in the ſame time as it produces 14½ when applied at the quarter.

Three quarts of water in a minute, laid on at the top of the wheel, turns it 60 times in 6 minutes and 45 ſeconds; four quarts ſet on at the quarter produces the ſame number of revolutions in the ſame time. And if the weight of the fly or the ratio of its revolutions to thoſe of the water-wheel be varied, yet the ratio of the quantities of water to produce the ſame effects in the ſame time remains the ſame.

Experiments on the Velocity of Wheels.

In theſe experiments, the effect is meaſured or eſtimated by the number of the revolutions which a fly makes in a given time: and although this is not the true meaſure of the power of the water, (part of it is exerted in turning the wheels), yet it is as much ſo as the revolutions of a mill-ſtone are the meaſure of the effect produced by the water; and that velocity which a wheel has when a mill-ſtone makes the greateſt number of revolutions in a given time, will certainly be conſidered as the beſt. It may be obſerved, that theſe experiments are confined to wheels on which the water acts by gravity only, and that the buckets are ſuppoſed to be large enough to retain the water till it comes near the bottom of the wheel: for if the buckets loſe part of the water as they deſcend, the wheel, in its preſent conſtruction, is not doing the moſt work.—In the following experiments, the quantity of water was always the ſame; the water-wheel made 60 revolutions, and ſometimes more; the time was obſerved, from which the number made in one minute was computed.

In order to vary the velocity of the water-wheel, it becomes neceſſary that the ratio of its revolutions, to thoſe of the fly, ſhould alſo vary, which is effected by changing ſome of the wheels.

Experiment 1. The water is applied at the top in this and all the following experiments under this head. The wheel makes 15.36 revolutions in a minute, the fly 21 for 1; and 15.36 × 21 = 322.56, the number of revolutions which the fly makes in a minute.

Experiment 2. The water-wheel turns 12.08 times in a minute, and the fly 28 for 1; hence the fly revolves 338.24 times per minute.

Experiment 3. The fly turns 44.6 times for the water-wheel once, which is turned 7.67 times in a minute; of conſequence, the fly revolves 342.08 times per minute.

Experiment 4. The fly makes 59.5 turns for the water-wheel 1, and the water-wheel turns 6.02 times in a minute; the fly, therefore, turns 358.19 times per minute.

But perhaps theſe four experiments, with others made with a different fly, may appear to more advantage as claſſed in the following table.

TABLE IV.

Experiments. 1.	Turns of the water-wheel per minute. 2.	Velocity of the water-wheel, per ſecond, in feet. 3.	Turns of the fly for one of the water-wheel. 4.	Turns of the fly per minute. 5.	Ratio of the quantity of water in the buckets. 6.	Quantity per minute to turn fly equally faſt. 7.	Turns of water-wheel per min. when fly turns equally faſt. 8.	Quantity on the wheel at once, when the fly has the ſame vel. 9.	Vel. of water-wheel in feet, when the motion of the fly is equable. 10.
1	15.36	1.433	21	322.5					
2	12.08	1.127	28	338.2					
3	7.67	.716	44.6	342					
4	6.02	.562	59.5	358.2					
5	36	3.36	8.75	315	1	4	36	1	3.36
6	28.125	2.62	13⅛	369.1	1.282	3.11	24	1.166	2.24
7	18.18	1.69	21	381.8	1.988	2.45	15	1.47	1.4
8	15.58	1.45	25	389.5	2.317	2.29		1.606	1.666
9	12.57	1.11	31½	395.6	2.863	2.14	10	1.926	.933
10	9.52	.88	42	400	3.812	2	7.4	2.4	.7
11	7.67	.71	52½	402.9	4.732	1.90	6	2.8	.56
12	6.4	.59	63	403.4	5.7	1.82	5	3.368	.466

The ſecond column ſhews the turns of the water-wheel per minute, as found by experiment.

The third is found from the ſecond: for example, in the firſt experiment, 15.36 multiplied by 5.6 feet, the circumference of the wheel, gives 86.016, the feet paſſed over in a minute, which divided by 60, gives the feet per ſecond, = 1.433.

The fourth column is known from the wheels made uſe of.

The fifth is the product of the ſecond multiplied by the fourth.

The ſixth ſhews the ratio of the quantity of water hanging upon the wheel at once, which is always inverſely as the velocity of the wheel: for inſtance, the wheel moving with a certain degree of velocity, every bucket receives a given quantity, but if the wheel moves twice as faſt, a bucket can receive but half the quantity; or if the wheel has half the firſt ſuppoſed velocity, a bucket will receive a double quantity. Otherwiſe, ſuppoſe the wheel moves 6 feet in a ſecond, and that a bucket receives 10 gallons; then, if it moves but 3 feet per ſecond, a bucket will receive 20 gallons, or if 2 feet per ſecond, 30 gallons, &c. In the fifth experiment, the wheel turns 36 times in a minute; in the ninth, but 12.57 times: hence, inverſely, As 36 : 1 portion of water :: 12.57 : 2.863 times the quantity.

By increaſing the velocity of the fly, compared with that of the water-wheel, we have, in the laſt experiment, near 6 times as much water upon the wheel at once, from a given ſtream, as there is in the fifth experiment, in which the fly is turned 315 times in a minute; but in the laſt, 403 times.

In the ſeventh column we have the quantity of water as found by experiments to produce 315 revolutions of the fly in one minute; and it appears, that by increaſing the velocity of the fly compared with that of the water-wheel, that a leſs quantity of water is ſufficient to produce the ſame effect, or the ſame number of revolutions of the fly: for in the laſt experiment the quantity of water is not half of that uſed in the fifth, though the effects are the ſame.

The eighth column contains the number of revolutions which the water-wheel makes in a minute, under the different changes of wheels in the communication of motion to the fly, when the fly itſelf makes the ſame number of revolutions per minute.

The ninth column exhibits the comparative quantity of water hanging on the wheel at once, when the fly moves equally faſt.

The tenth is the velocity of the water-wheel, in feet, when the fly has the ſame velocity in all the experiments.

The five following experiments were made with a wheel of 11 inches diameter.

TABLE V.

Experiments.	Turns of the water-wheel per minute.	Velocity of the water-wheel, per ſecond, in feet.	Turns of the fly for one of the water-wheel.	Revolutions of the fly per minute.
13	14	.677	21	296
14	9.6	.464	31½	302.4
15	7.31	.353	42	307
16	5.9	.285	52½	309.75
17	4.94	.238	63	311.55

The ſpeed of the water-wheel being diminiſhed, the revolutions made by the fly in a given time are increaſed in number, both with the large and ſmall wheel. It is true they are not increaſed in the ſame ratio when the velocity is much diminiſhed, as in the eleventh and twelfth in Table firſt, and again in the ſixteenth and ſeventeenth in Table ſecond, but this is owing to the buckets being overloaded, ſo that they loſt a part of their water too ſoon, and of conſequence the effect in the two laſt experiments is not much increaſed. But if the buckets retain their water till they come near the bottom, it appears, that the ſlower the water-wheel moves, the greater its effect will be, provided the motion continues to be equable. See Prob. XXXVIII. Part II.

This may alſo be proved by raiſing weight by a cord coiled round the axle of the water-wheel.

The following table contains ſeven experiments, in which the velocity of the water-wheel is gradually diminiſhed from the firſt to the laſt; but the effect in the laſt is the greateſt.

TABLE VI.

Experiments.	Weight.	Time of riſing thro' 6 feet.	Velocity, or feet per ſecond.	Effect produced.
1	½lb.	57.5″	.1043	5.21
2	1	62.4	.0961	9.61
3	1½	69.5	.0863	12.94
4	2	75.5	.0795	15.90
5	3	93	.0645	19.35
6	4	117	.0512	20.48
7	5	142	.0422	21.10

To find the velocity per ſecond. As 57.5″ : 6 feet :: 1″ : .1043 feet; and As 62.4″ : 6 feet :: 1″ : .0961 feet, and ſo for the others. The effect is obtained by multiplying the velocity thus found by the weight raiſed, as in the firſt experiment, ½ lb. multiplied by .1043, gives .0521, the effect; and in the ſecond, 1 lb. multiplied by .0961 = .0961, and in the ſame manner the effect in all the other experiments is obtained. In the table, the effect thus found is multiplied by 100, or the decimal point is removed two figures, but the ratio is the ſame.

If it was required to find the weight raiſed in a given time, ſuppoſe 1 hour, it would be, *As the time of riſing through 6 feet is to the weight raiſed,* *ſo is* 1 *hour to the number of times which the ſaid weight would be raiſed through the given ſpace.*

For example, As 57.5″ : ½ lb. :: 3600″ (1 hour) : 31.304, pounds per hour. In the third experiment, As 69.5″ : 1.5 lb. :: 3600″ : 77.7, pounds per hour. In the ſixth experiment, As 117″ : 4 lb. :: 3600″ : 123.08, pounds per hour. In the ſeventh experiment, As 142″ : 5 lb. :: 3600″ : 126.7, pounds per hour.

Which experiments ſo far, in conjunction with thoſe made with the fly, prove that by diminiſhing the velocity of the water-wheel, we increaſe the power.

On the Velocity communicated to a Wheel, by different quantities of Water.

See page 17,—where it appears that if a given ſtream produces a certain velocity in the wheel, a ſtream eight times as large will be required to communicate double that velocity, or to turn the wheel twice round in the ſame time. And for any other quantity of water which may be applied, it will be, *As the cube root of the preſent quantity is to one degree of velocity, ſo is the cube root of the other quantity to the velocity it will produce.*

The truth of the propoſition will appear by the following experiments.

FIG. XXX.

Pieces of braſs were ſcrewed upon the end of the cock, and perforated, by trial, till the aperture was large enough to deliver the exact quantity required: for inſtance, a pint, a quart, two quarts, &c. per minute, and which quantity was always applied in the ſame manner.

Experiment 1. One pint of water per minute turns the water-wheel 60 times in 13 minutes and 33 ſeconds.

Experiment 2. Two pints per minute turns it 60 times in 10 minutes and 42 ſeconds.

Experiment 3. Four pints per minute produces 60 turns in 8 minutes and 27 ſeconds.

Experiment 4. Six pints per minute produces 60 turns in 7 minutes and 18 ſeconds.

Experiment 5. Eight pints per minute produces 60 turns in 6 minutes and 18 ſeconds.

In the following table we have, in the firſt column, the quantity of water applied; in the ſecond, the cube root of that quantity; in the third, the comparative velocities as found by the experiments; in the fourth, the number of turns the water-wheel made in a minute.

TABLE VII.

Experiments.	Water per min.	Cube root of the quantity of water.	Velocity of the wheel.	Turns per min.
1	1	1	1	4.42
2	2	1.2599	1.267	5.6
3	4	1.5874	1.606	7.1
4	6	1.8171	1.859	8.219
5	8	2	2.15	9.52

The number of turns per minute is obtained by direct proportion; for, in Experiment 1, As 13′ 33″ : 60 revolutions :: 60″ : 4.42 revolutions, and ſo for the reſt.

The velocity is directly as the number of turns made in a given time; and if in the firſt experiment it is repreſented by 1, in the ſecond it will be 1.267; for As 4.42 : 1 :: 5.6 : 1.267, and As 4.42 : 1 :: 7.1 : 1.606, &c.

In the fifth experiment, the quantity per minute is eight times as much as in the firſt; but it may be obſerved, that the load or weight upon the wheel is only four times as much as in the firſt; for as the wheel moves twice as faſt, a bucket a bucket can only receive half as much water as it would do with the velocity it had in the firſt experiment.

ON THE SIZE OF WHEELS.

The fall being given, or the greateſt altitude at which a given ſtream can be applied to a wheel, it becomes neceſſary, both on account of expence and utility, to enquire what the diameter of the wheel ought to be.—This may be determined either by theory or experiment.

1. Let us ſuppoſe two wheels equally heavy, the diameter of one 10, and the other 20 feet, and that the weight is chiefly in the circumference, or as much ſo in one as in the other.

2. Let the ſame power be applied to the circumference of each wheel, and each circumference will have the ſame velocity; hence, the time of a revolution will be as the diameters directly, or in the preſent caſe as 1 to 2, or the 10 feet wheel will be turned twice for the 20 feet once, as already demonſtrated in Prob. XIV, Part II, and proved by experiment in No. 12 and 16 in Table II, page 128. In both experiments the power is 4 oz., and the weight of the fly or wheel is the ſame; but in No. 12 the

radius is 8 inches, and the power is applied at the diſtance of 2 inches; in No. 16 its radius is 4 inches, and the power is applied at 1 inch: hence the ratio of the effects produced is exactly the ſame as if it had been applied to the circumference of the two wheels; and as their weights are equal, and they move equally faſt, the effects are alſo equal: but their revolutions are not performed in the ſame time; for, in 26 ſeconds, one revolves 61 times, and the other 122, or the revolutions are performed in times which are as 1 to 2, as above.

3. The power being employed to turn a ſingle wheel, the velocity of its circumference will vary with the weight of the wheel, and of conſequence the time in which it revolves: for if one of the above wheels is made twice as heavy, it will be twice as long in making one revolution as it was before.

4. If the power is a ſtream of water, this will vary with the velocity of the wheel. Suppoſe a given ſtream to turn the wheel with two degrees of velocity, then if the velocity of the wheel is but one, there will be twice as much water upon it, or in the buckets, as when it had two degrees of velocity: hence, if a wheel of a given weight has a velocity of 4 feet per ſecond, one 4 times as heavy would be cauſed to move 2 feet per

ſecond by the ſame ſtream (ſee page 112, and 124), not conſidering the water as part of the maſs to be moved.

5. Let us ſuppoſe a ſtream of water applied at the top of a wheel 10 feet in diameter, and that it turns the wheel in 6 ſeconds; let the ſame ſtream be applied at the center of another wheel 20 feet in diameter; then it is evident, that if every bucket receives as much water as before, that the large wheel will be twice as long in making one revolution as the ſmall wheel, and therefore by the ſame cog-wheel, &c. cannot do the ſame work; and if we increaſe the cog-wheel in the ſame ratio as the water-wheel, we obtain no power by making the wheel larger.

The truth of the above is confirmed by the following experiments.

Experiment 1. A given ſtream is applied exactly at the center of a wheel 20 inches in diameter; it makes 60 revolutions in 4 minutes and 21 ſeconds.

Experiment 2. The large wheel is removed, (the cog-wheel, fly, &c. remain the ſame) and one of 10 inches in diameter is fixed in its place; the ſame quantity of water is applied at the top; and it makes 60 revolutions in 4 minutes and 15 ſeconds.

Experiment 3. The water is applied at the center of the great wheel, and in addition to the fly, one pound is ſuſpended from the axle, which is raiſed through 5 feet in 59½ ſeconds. The large wheel being removed, and the ſmall one fixed in its place, every thing elſe remaining the ſame, the ſame weight is raiſed through the ſame ſpace in 55 ſeconds.

In the laſt experiment, if the great wheel produces the ſame effect as the ſmall wheel in the ſame time, the circumference muſt move twice as faſt, by which the power would be diminiſhed. See page 137, &c.

Experiment 4. A cylinder is fixed upon the axle, which has the ſame ratio to the diameter of the large wheel as the axle itſelf has to that of the ſmall one, or the axle is 3 inches diameter, and the cylinder 6 inches; the water is applied at the center of the great wheel, and the weight is raiſed in 56 ſeconds.

In the above experiments, except the third, there ought to be no difference in the effects produced by the two wheels, but what ariſes from the difference of the weight of the ſaid wheels and their velocities; hence the greater effects produced by the ſmall wheel is owing to its requiring a leſs force to turn it, independent of the reſt of the machinery, than what is neceſſary to turn the large wheel, or to cauſe it to revolve in the ſame time (ſee Prob. XIV, Part II). For if the wheels were of the ſame weight, and the ſame power applied to the circumference of each, the ſmall wheel would make two revolutions for the large wheel one.

ON THE FALL OF WATER.

BY the *fall*, is underſtood, the perpendicular altitude, meaſured from the bottom of the wheel to the ſurface of the dam, which, when the water iſſues upon the wheel at ſome diſtance below the ſurface, is frequently diſtinguiſhed by *head* and *fall*.

The effect produced by a given ſtream in falling through a given ſpace, if compared with a weight, will be directly as that ſpace; but if we meaſure it by the velocity communicated to the wheel, it will be as the ſquare root of the ſpace deſcended through, agreeable to the laws of falling bodies.

Experiment 1. A given ſtream is applied to a wheel, at the center; the revolutions per minute are 38.5, Experiment 2. The ſame ſtream applied at the top, turns the ſame wheel 57 times in a minute.

If in the firſt experiment the fall is called 1, in the ſecond it will be 2. Then, As $\sqrt{1} : \sqrt{2}$ :: 38.5 : 54.4, which are in the ſame ratio as the ſquare roots of the ſpaces fallen through, and near the obſerved velocity.

In the following experiments a fly is connected with the water-wheel.

Experiment 3. The water is applied at the center, the wheel revolves 13.03 times in one minute.

Experiment 4. The water is applied at the vertex of the wheel, and it revolves 18.2 times per minute.

And As 13.03 : 18.2 :: $\sqrt{1} : \sqrt{2}$, nearly.

From the above we infer, that the circumferences of wheels of different ſizes may move with velocities which are as the ſquare roots of their diameters without diſadvantage, compared one with another, the water in all being applied at the top of the wheel. For the velocity of falling water at the bottom or end of the fall is as the time, or as the ſquare root of the ſpace fallen through: for example,

ample, let the fall be 4 feet; then, As $\sqrt{16} : 1'' :: \sqrt{4} : \frac{1}{2}''$, the time of falling through 4 feet. Again, let the fall be 9 feet; then, As $\sqrt{16} : 1'' :: \sqrt{9} : \frac{3}{4}''$, and ſo for any other ſpace, as in the following table, where it appears that water will fall through 1 foot in a quarter of a ſecond, through 4 feet in half a ſecond, through 9 feet in three quarters of a ſecond, and through 16 feet in one ſecond. And if a wheel 4 feet in diameter moved as faſt as the water, it could not revolve in leſs than 1.5 ſeconds, neither could a wheel of 16 feet diameter revolve in leſs than 3 ſeconds; but though it is impoſſible for a wheel to move as faſt as the ſtream which turns it, yet, if their velocities bear the ſame ratio to the time of the fall through their diameters, the wheel 16 feet in diameter may move twice as faſt as that of 4 feet.

TABLE VIII.

Height of the fall, in feet.	Time of falling, in ſeconds.	Height of the fall, in feet.	Time of falling, in ſeconds.
1	.25	14	.935
2	.352	16	1
3	.432	20	1.117
4	.5	24	1.22
5	.557	25	1.25
6	.612	30	1.37
7	.666	36	1.5
8	.706	40	1.58
9	.75	45	1.67
10	.79	50	1.76
12	.864		

When different ſtreams of water produce equal effects, the quantities muſt be inverſely as the fall. For inſtance, if 6 cubic feet in falling through 4 feet produces a certain effect, 3 cubic feet in falling through 8 feet would produce an equal effect; or the quantity being the ſame, the effect will be as the ſpace fallen through.

The reſiſtance afforded by mill-ſtones, and other kinds of machinery, cannot be computed from firſt principles. We can only aſcertain it by obſervations: as for example,

In a corn-mill,

Suppoſe 4 cubic feet per ſecond, applied at the top of a wheel 12 feet high, will turn the mill-ſtone a given number of times per minute; what is the reſiſtance of the whole, at the circumference of the water-wheel?

Let the wheel revolve in 8 ſeconds; then will the buckets contain as much water at once as is delivered in 4 ſeconds, or half a revolution, viz. 16 cubic feet, or 1000 lb. But as this weight is equally diſperſed over one half the circumference of the wheel, the weight acting at the extremity of the horizontal arm will be leſs; for the diameter being perpendicular, the center of gravity of the circular arch will be .6366 of the radius, from the axis; therefore, from the principle of the lever, .6366 multiplied by the whole weight 1000 lb., gives 636.6, the weight at the circumference, or at the end of the horizontal arm.

EXAMPLE II.

Suppoſe the wheel revolves in 10 ſeconds, every thing elſe remaining the ſame.

In this caſe, 20 cubic feet, or 1250 lb. of water, is contained in the buckets, which multiplied by .6366, gives 795.7 lb. for the power at the end of the horizontal arm.

In the above examples I have ſuppoſed that the buckets contain their water till they come to the bottom; and if the wheel moves from 3 to 4 feet per ſecond, the buckets may be made of ſuch a form as to carry the water quite to the bottom, provided they are large enough, when at the center, to hold near twice as much water as is neceſſary to do the work; and if the wheel moves faſter, they will carry the water too far.

The effects of impulſe and gravity compared.

From the experiments contained in the Table, page 114, it appears, that the impulſive force of ſtream, is to the abſolute weight of the column which impels it, as 16 to 10, or as 8 to 5, nearly; from which we may eaſily compare the impulſe with the gravity of a given quantity.

For example,

Let the head be 4 feet, and the area of the ſluice 1 inch; then will the velocity be 10.8 feet per ſecond: that is, the ſquare root of the depth multiplied by 5.4.

The ſolid content of the column which impels the ſtream is 4 feet multiplied by 1 inch = 48 cubic inches, and its weight is 27.77 oz. The quantity per ſecond is 10.8 feet long, and 1 inch area, = 129.6 cubic inches, or 75 oz.

The velocity of the wheel is one-third the velocity of the ſtream, = 3.6 feet per ſecond, (ſee page 29). And As 5 : 8 :: 27.77 oz., the whole weight of the column, : 44.44 oz., the impulſive force upon an object at reſt. But, the quantity being the ſame, the force is as the ſquare of the velocity with which it ſtrikes the wheel, in this caſe, equal to the difference between 10.8, the velocity of the ſtream, and 3.6, the velocity of the wheel, that is, 7.2 feet per ſecond. Hence, As $\overline{10.8}|^2 : \overline{7.2}|^2 :: 44.44 : 19.6$ oz., the impelling force conſtantly acting upon the wheel. Or if the velocity of the wheel ſhould have any other ratio to the velocity of the ſtream, ſuppoſe 1 : 2, or that the wheel moves half as faſt as the ſtream; then, As $\overline{10.8}^2 : \overline{5.4}^2 ::$ 44.44 oz. : 11.11 oz., the force acting upon the wheel.

In the overſhot.

Let the ſame quantity of water be thrown upon the top of a bucket-wheel 4 feet diameter.

The quantity per ſecond is 75 oz. If the wheel moves as faſt as before, which we will ſuppoſe for the ſake of the contraſt, it will revolve in 3.48 ſeconds, and will at once be loaded with as much water as is delivered in 1.74 ſeconds, viz. in the time of half a revolution; therefore 75 oz. × 1.74 = 130.5 oz., the weight conſtantly acting on the overſhot-wheel; but as it is not all gravitating at the end of the horizontal arm, but diſperſed through a ſemicircle, we muſt, as before obſerved, multiply the whole weight, 130.5 oz., by .6366, the diſtance of the center of gravity in the ſemicircle from the center of the circle, and it gives 83 oz., equal to the effect of all the water at the end of the horizontal arm, which is more than 4 times the impulſe of the ſame quantity.

EXAMPLE II.

Let the head be 9 feet, and area of the ſluice 1 inch.

Then, the ſquare root of 9×5.4, gives the velocity per ſecond, $= 16.2$ feet; and 16.2×12, gives the cubic inches per ſecond, $= 194.4$, which multiplied by .5787 oz., the weight of 1 inch is $= 112.5$ oz.; weight of a column 9 feet long and 1 inch ſquare $= 62.3$ oz. Then, As 5 : 8 :: 62.3 oz. : 99.68 oz., the impulſe upon an object at reſt.

The velocity of the wheel $= \frac{16.2}{3} = 5.4$ feet per ſecond, which taken from the velocity of the ſtream, leaves 10.8, the velocity with which the ſtream ſtrikes the wheel. Then As $\overline{16.2)}^2 : \overline{10.8)}^2$ :: 99.68 oz. : 44.3 oz., the whole force of the ſtream upon the wheel.

In an overſhot.

The circumference of the wheel divided by the velocity, gives $(\frac{28.27}{5.4})$ 5.2 ſeconds for the time of a revolution, half of which multiplied by the weight per ſecond, gives 292.5 oz. for the weight of water upon the wheel, when it moves with the above velocity, which being multiplied by .6366, gives 186.2 oz., the force exerted in the direction of a tangent to turn the wheel.

Many have expreſſed their ſurpriſe that the impulſe of a given ſtream ſhould be leſs than its gravity: here I have admitted the impulſe to be almoſt double the gravity, but the reaſon why its effect is leſs does not ſeem very ſurpriſing.

OBSERVATIONS ON BUCKETS AND FLOATS.

If the floats of a wheel are fitted to move in a channel of the ſame curvature (provided there was no loſs of water), the effect of the water would be the ſame as in a bucket-wheel.

Demonſtration.

FIG. XXIX.

Let CBB be part of a water-wheel; and let the weight of the water in the bucket $abd = w$; the ſine of the angle $\text{BC}d = \text{CA} = s$; CB, the radius, $= r$. Then will $\frac{ws}{r}$ be the force exerted at B to turn the wheel in the direction of a tangent.

Next, if B$bd$B repreſent a curved channel in which a float-wheel moves, then the water abd will be ſupported by the channel db and float ad. The ſine of the angle which the channel db makes with the horizon will be equal to that of the angle BCd, and the preſſure which the float ſuſtains will be obtained from the property of the inclined plane, viz. *As the radius is to the ſine of the angle of elevation, ſo is the whole weight* *to that part which the float ſupports*; or, As $r : s :: w : \frac{ws}{r}$, the ſame as in the bucket-wheel.

EXAMPLE I.

Let a bucket containing 100 lb. of water be at the diſtance of 30 degrees from the bottom, to determine the force with which it acts on the wheel, in the direction of a tangent?

The natural ſine of 30° is .5, the radius being 1; therefore, $\frac{100 \times .5}{1} = 50$ lb., the whole force of the water whether confined in a bucket or in the channel.

EXAMPLE II.

Suppoſe the bucket or float at the diſtance of 50 degrees from the perpendicular?

The natural ſine of 50° is .766, which multiplied by 100, the weight of the water, gives 76.6 lb. for the force; and ſo for any other.

From which it appears that there is no difference, except the loſs of water in the float-wheel, for, however well it may be executed, ſome water will paſs; nevertheleſs, when the ſtream is large, a wheel with floats may be found more convenient than one with buckets. For example, ſuppoſe a ſtream flowing over a dam to be 6 feet wide, and 1 foot deep; the quantity per ſecond would be 21 cubic feet (ſee page 108). Suppoſe 3 buckets or floats paſs the ſtream in 1 ſecond, there will be 7 cubic feet for every bucket or float. In a float-wheel there ought not to be more than ſix-eighths of the ſpace filled with water; and in a bucket-wheel little more than one-half. The buckets, therefore, ought to contain about 14 cubic feet each, when at the end of the horizontal arm; and the ſpace between the floats ought to be large enough to contain about 10 cubic feet; in which caſe, if the floats are 15 inches diſtant, they ought to be 16 inches deep. But for the bucket-wheel, the ſhrouds would be near 2 feet deep if large enough to contain the water. In which caſe, a float-wheel ought to be preferred: and if a wheel is occaſionally in back-water, floats are not ſo much retarded as buckets.

Experiments with wheels acted on by impulſe only.

In addition to the experiments made on bucket-wheels, in which the water produces the effect by its *gravity*, I ſhall add a few made with float-wheels, on which the water acts by *impulſe* only. If the wheel is confined in a channel, after the water has ſtruck the float, it is retarded, and heaped againſt the float; in which caſe, the water is acting by impulſe and gravity combined. In order to aſcertain, as near as poſſible, the true effect of the impulſe, the following experiments were made. The theory (page 29) aſſigns no more than a comparative force to the water; or, if a given quantity iſſues with a given velocity, the effect produced ſhould be as the ſquare of the velocity with which it ſtrikes the wheel, and will of conſequence vary with the velocity of the wheel; and the velocity of the wheel will alſo vary with the reſiſtance. The effect of the ſtream will be greateſt when it ſtrikes the floats in a perpendicular direction, and will diminiſh as the obliquity increaſes; and as the greateſt force is exerted on the firſt float, it was neceſſary, in theſe experiments, to fix the floats a little inclined to the circumference of the wheel, or inſtead of the common direction c*s*, (Fig. XXXI) to direct them to the circumference of a circle *bb*, in the direction *bs*, ſo that the direction of the ſtream may be nearly perpendicular to the firſt float *sb*, which is acted upon till another intercepts the ſtream, during which time its poſition is conſtantly changing.

The velocity of the ſtream was 9.3 feet per ſecond. The weight of water per minute was 10 lb. The diameter of one wheel is 12 inches; but I take the circumference of that part of the floats which was ſtruck by the water to be 2.8

feet, and the circumference of the other at 4.4 feet. The axle was truly turned, and ſupported on friction-wheels, and by 41 revolutions raiſed a weight through 5 feet.

Experiments with the 12 inch wheel.

TABLE IX.

Experiments.	Weight raiſed, in ounces. 1.	Time of riſing 5 feet. 2.	Velocity of the weight, in feet, per minute. 3.	Velocity of the wheel. 4.	Vel. with which the wheel is ſtruck. 5.	Effect, by experiment. 6.	Effect, computed. 7.	Ratio of experimental effects. 8.
1	7	30	10	3.83	5.47	70	113.6	114.3
2	8	33	9.09	3.45	5.85	72.72	118.05	118.7
3	8½	35	8.57	3.3	6	72.8	118.8	118.9
4	9	37	8.11	3.1	6.2	72.9	119.16	119.16
5	9½	39½	7.6	2.91	6.39	72.2	118.76	118
6	10	46	6.52	2.5	6.8	65.2	115.5	106.5
7	11	56	5.35	2.05	7.25	58.8	107.14	96.1

Experiments with the 18 inch wheel.

TABLE X.

1	9	41½	7.22	4.34	4.96	64.98	106.61	105.9
2	10½	45½	6.6	3.95	5.35	69.3	113.01	112.77
3	11	47	6.38	3.83	5.47	70.18	114.47	114.4
4	12½	52	5.77	3.47	5.83	72.12	117.9	117.5
5	13	54	5.55	3.34	5.96	72.22	118.6	117.65
6	14½	66	4.54	2.73	6.57	65.83	117.79	107.3

While the 12 inch wheel makes 41 revolutions, the part of the float which is ſtruck paſſes over 115 feet, and the ſame part of the 18 inch wheel moves 180.4 feet: hence the velocity per ſecond is found as follows; in experiment 1, As 30″ : 115 feet :: 1″ : 3.83 feet, &c.

The velocity with which the ſtream ſtrikes the wheel is alſo known, by ſubtracting the velocity of the wheel from that of the ſtream; as in the firſt experiment, 3.83 taken from 9.3, leaves 5.47 for the velocity with which the wheel is ſtruck.

Alſo, As 30″ : 5 feet :: 60″ : 10 feet, the ſpace paſſed over by the weight in one minute; which ſpace multiplied by the weight, gives the effect contained in the ſixth column.

The greateſt effect produced by the 12 inch wheel is in the fourth experiment, where 9 oz. is raiſed through 8.11 feet in a minute; their product is 72.9 for the maximum.

But the maximum in the 18 inch wheel is in the fifth experiment, where 13 oz. is raiſed 5.55 feet in a minute, and their product is 72.2, which is not quite ſo much as that of the leſs wheel; and, indeed, the difference would have been greater if the wheels had been equally heavy, but the larger wheel weighs 12 oz., and the ſmaller 1 lb. 14 oz. It may alſo be obſerved, that the larger wheel is in this experiment rather ſhort of the maximum, its velocity is 3.34 inſtead of 3.1, and in the next experiment it moves too ſlow, or is decreaſing in power.

To compare the power with the effect.—We have 10 lb. or 160 oz. of water falling through 2.2 feet in one minute; their product, 352, is the power; the effect is 73, or $\frac{73}{352}$ of the power $= \frac{.206}{1000}$. But in the overſhot (ſee page 143) the power is to the effect as 18 : 12.675, or as 1 : .704, or the effect is $\frac{704}{1000}$ of the power, which is 3.4 times as much as by impulſe, though in theſe experiments we have the advantage of friction-wheels, which were wanting in the experiments with the overſhot; beſides, the overſhot was encumbered with a fly, without which the motion would not have been regular.

We may further obſerve, that the effects obtained by multiplying the ſquare of the velocity with which the water ſtrikes the wheel, by the velocity of the wheel, as contained in the ſeventh column of the table, agrees, as near as can be expected, with the experiments, till we come at a maximum; after which, the effect, as found by experiment, varies from that computed by the ſquare of the velocity; and before this, the weights raiſed are nearly as the ſquares of the velocities with which the wheel is ſtruck. To compare which, the eighth column is added, and is found by proportion. For as the ſeventh is only comparative, if we ſay, beginning with the maximum, Experiment 4th, Table 9th, As 72.9, the real effect, is to 119.16, the relative effect, ſo is 72.8 to 118.9, the relative, &c.; by which proceſs, the eighth column is formed from the experiments, in order to compare them with the theory.

ON UNIFORM VELOCITY.

A wheel turned by the weight or impulſe of water has its velocity increaſed till the reſiſtance becomes equal to the accelerating force. The reſiſtance may be of various kinds; and in many caſes it may be not only difficult, but impoſſible to bring it into a computation. See page 154. In the preſent obſervations, I ſhall conſider the reſiſtance as ariſing from the weight of the wheel, and a weight to be raiſed by a cord fixed to its axle.

The following experiments were made with a bucket-wheel 20 inches in diameter, and ſupported on friction-wheels: the circumference of its axle 9 inches; the whole weight of the wheel and axle 8lb.; the center of gyration is at 8.4 inches; and the reſiſtance, at the diſtance of 10 inches, will be about 92 oz.

TABLE X.

Number.	Weight raiſed, in ounces.	Reſiſtance of the wheel.	The whole reſiſtanse.	Time of riſing 5 feet.	Velocity of the wheel, in feet, per ſecond.	Velocity, by computation.	Ounces of water on the wheel.
	m	q	p	t	v		
1	4.5	3.5 oz.	8 oz.	$13\frac{1}{2}''$	2.48	2.89	2.68
2	8.5	2.35	10.85	$16\frac{1}{2}$	2.03	2.12	3.28
3	12.5	1.54	14.04	$21\frac{1}{2}$	1.55	1.63	4.3
4	16.5	.98	17.48	27	1.24	1.30	5.37
5	20.5	.7	21.2	32	1.05	1.07	6.35
6	24.5	.52	25.02	37	.90	.92	7.4

In the firſt experiment, the water-wheel revolves 6.7 times in $13\frac{1}{2}$ ſeconds, and raiſes $4\frac{1}{2}$ oz. through 5 feet; the velocity of the circumference is 2.48 feet per ſecond.

In the ſecond experiment, the wheel makes the ſame number of turns in $16\frac{1}{2}$ ſeconds, and raiſes $8\frac{1}{2}$ oz. through the ſame ſpace, &c.

We are next to enquire, what weights applied to the axle would turn the wheel, and deſcend through 5 feet in $13\frac{1}{2}$, $16\frac{1}{2}$, $21\frac{1}{2}$, &c. ſeconds. (See Theo. 4, page 124). We have given,

$F = 5$ feet, the ſpace paſſed over by the weight.
$m =$ the weight ſuſpended from the axle.
$w = 92$ oz., the weight of the wheel multiplied by the ſquare of the center of gyration, and divided by the ſquare of its radius.

$s =$

s = 10 inches, the radius of the wheel.
r = 1.42 inches, the radius of the axle.
t = the time of deſcent.
d = 16, the feet fallen through by gravity, in one ſecond.

Then we have the weight $m = \frac{Fws^2}{dr^2t^2 - Fr^2}$ = 7.02 oz., when $t = 13\frac{1}{2}$ ſeconds; but when $t = 16\frac{1}{2}$, $m = 4.7$ oz. &c. Now, if 7 oz. falls through 5 feet with an accelerated velocity, by the laws of falling bodies, it would, at the end of the time, have acquired a velocity which, if equable, would carry it over 10 feet in the ſame time. And the whole effect is equal to what half the weight would produce in paſſing over the ſame ſpace in the ſame time, with an uniform velocity. Half the weights found by the above theorem are placed in the ſecond column, under q, which being added to the weight raiſed, gives the whole reſiſtance, as in the next column.

To find the quantity of water upon the wheel.

If the quantity delivered in one ſecond, be multiplied by the ſeconds in which the wheel makes half a revolution, the product will be the whole quantity upon the wheel.

Or, if half the circumference, in feet, is multiplied by the quantity per ſecond, and divided by the velocity, the quotient will be the quantity upon the wheel.

Let $Q =$ the quantity per ſecond.
$c =$ half the circumference.
$v =$ the velocity of the circumference.
$m + q = p =$ the ſum of the weight and reſiſtance.

Then will $\frac{Qc}{v} =$ the quantity upon the wheel, which quantity will vary with the velocity.

$\frac{pr}{s} =$ the power applied at the circumference, which will balance $m + q$, when in motion. Hence, $\frac{Qc}{v} - \frac{pr}{s} = \frac{Qcs - vpr}{vs} =$ the efficacious moving power; and the reſiſtance is $\frac{Qc}{v} + w + \frac{pr^2}{s^2} = \frac{Qcs^2 + wvs^2 + pvr^2}{vs^2}$; by which divide the moving power, and we have $\frac{Qcs^2 - vprs}{Qcs^2 + wvs^2 + pvr^2}$ for the accelerating force, which, when the motion becomes uniform, muſt be equal to the retarding force, which will be expreſſed by $\frac{pr}{s} \div w + \frac{pr^2}{s^2} = \frac{prs}{s^2w + r^2p}$; from which we find $v = \frac{Qcs}{2pr} \times \frac{s^2w + pr^2 - prs}{s^2w + pr^2} = \frac{Qcs}{2pr} \times \overline{1 - \frac{prs}{ws^2 + pr^2}}$.

The value of v found by the theory, may next be compared with what it is in the experiment, in which we have given $Q = 2\frac{2}{3}$ oz., $C = 2\frac{1}{2}$ feet, the reſt of the notation as above, except p, which is different in every experiment.

In the firſt experiment $p = 8$, and $\frac{QCs}{2pr} = \frac{66.66}{22.72} = 2.934$; and $\overline{1 - \frac{prs}{ws^2 + pr^2}} = .987$, which multiplied by 2.934, gives $2.897 = v$.

In the ſecond, $p = 10.85$, and $v = 2.12$, as in the table; in which it appears, that the velocity found by computation is, in every experiment, ſomething too much: but the difference is leſs than could be expected, when we conſider the degree of accuracy which is required, in the quantity of water, the weight, diameter, inertia, friction, &c.

When half the buckets are equally loaded with water, and the wheel at reſt; if the whole weight of the water is multiplied by .6366, the product is the weight which, if ſuſpended from the end of the oppoſite horizontal radius, would balance the whole. But, as near as I can diſcover by experiment, when the wheel is in motion, the effect is nearly the ſame as if all the water in the buckets gravitated on the end of the horizontal diameter. If this is true, it may be owing, in part, to the velocity which the water has when thrown upon the wheel, and to the centrifugal force, by which it endeavours to move in a tangent to the wheel. But, in making computations, to be on the ſafe ſide, it will be better to multiply the quantity by .6366.

When the reſiſtance is irregular, as in forges, tilts, ſaw-mills, rolling-mills, &c. it becomes neceſſary to add a fly, in order to preſerve, as near as may be, an equable motion, and to avoid ſudden ſhocks in the work.

Where there is convenience, a large fly ought to be preferred. For if a fly of 10, and another of 20 feet diameter, revolve in the ſame time, the ſmaller muſt be 4 times as heavy as the other, to produce the ſame effect.

In page 143, 5 lb. is raiſed 6 feet in 142 ſeconds, or through 21 inches, the diameter of the wheel, in 42 ſeconds, in which time 7 lb. of water deſcends through the ſame ſpace; hence, two-ſevenths of the whole is employed in overcoming the friction, reſiſtance, &c., and the power is to the effect as 7 to 5.

In the ſixth experiment of the laſt table, 24.5 oz. is raiſed through 5 feet in 37 ſeconds, or through the diameter of the wheel in 12.95 ſeconds, The value of v found by the theory, may next be compared with what it is in the experiment, in which we have given $Q = 2\frac{2}{3}$ oz., $c = 2\frac{1}{2}$ feet, the reſt of the notation as above, except p, which is different in every experiment.

In the firſt experiment $p = 8$, and $\frac{QCs}{2pr} = \frac{66.66}{22.72} = 2.934$; and $\overline{1 - \frac{prs}{ws^2 + pr^2}} = .987$, which multiplied by 2.934, gives $2.897 = v$.

In the ſecond, $p = 10.85$, and $v = 2.12$, as in the table; in which it appears, that the velocity found by computation is, in every experiment, ſomething too much: but the difference is leſs than could be expected, when we conſider the degree of accuracy which is required, in the quantity of water, the weight, diameter, inertia, friction, &c.

When half the buckets are equally loaded with water, and the wheel at reſt; if the whole weight of the water is multiplied by .6366, the product is the weight which, if ſuſpended from the end of the oppoſite horizontal radius, would balance the whole. But, as near as I can diſcover by experiment, when the wheel is in motion, the effect is nearly the ſame as if all the water in the buckets gravitated on the end of the horizontal diameter. If this is true, it may be owing, in part, to the velocity which the water has when thrown upon the wheel, and to the centrifugal force, by which it endeavours to move in a tangent to the wheel. But, in making computations, to be on the ſafe ſide, it will be better to multiply the quantity by .6366.

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In the ſixth experiment of the laſt table, 24.5 oz. is raiſed through 5 feet in 37 ſeconds, or through the diameter of the wheel in 12.95 ſeconds,

conds, in which time 34.5328 oz. of water deſcends through the ſame ſpace; and As 24.5 : 34.5328 :: 5 : 7.03, ſo is the effect to the power, the ſame as above, nearly. If more water is applied, the velocity is increaſed; but in the ſame wheel, the ratio of the power to the effect will always be nearly the ſame, at a maximum.

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FINIS.

ERRATA.

Page 4, *line* 7, *for* $\frac{V}{SF}$ *read* $\frac{V^2}{8F}$; *p*. 5, *l*. 10, *for* $\frac{FD}{D}$ *r*. $\frac{Fd}{D}$; *p*. 7, *l*. 10, *eraſe* per ſecond; *p*. 7, *l*. 11, *r*. 15.279 feet per ſecond; *p*. 8, *l*. 8, *for* × 9 *r*. × *q*; *p*. 9, *l*. 17, *for* 2.44 *r*. 2.59, the diameter of the circle of gyration; *p*. 13, *l*. 17, *for* and *r*. then; *p*. 26, *l*. 17, *for* $w + ax$ *r*. $wb + ax$; *p*. 51, *l*. 8, *for* $ap + aw$ *r*. $aP + aW$; *p*. 51, *l*. 13, *for* $\frac{mn^2}{mn^2}$ &c. *r*. $\frac{mn^3}{mn^2}$ &c.; *p*. 53, *l*. 20, *for* $\frac{64}{6}$ *r*. $\frac{64}{9}$, in the denominator; *p*. 62, in problem xxx, *add* $W = 78$; *p*. 63, *l*. 15, *for* $4q = 2x$ *r*. $4q \times 2x$; *p*. 65, *l*. 24, *for* $\frac{Pz - W}{y^2m + Pz^2 + W}$ *r*. $\frac{P - Wz}{my^2 + Wz^2 + P}$; *p*. 67, *l*. 7, *for* 2.63 *r*. .263; *p*. 69, *l*. 7, *for* $\frac{1}{a}$ *r*. $\frac{1}{a^2}$; *p*. 72, *l*. 12, *for* venculum *r*. vinculum.

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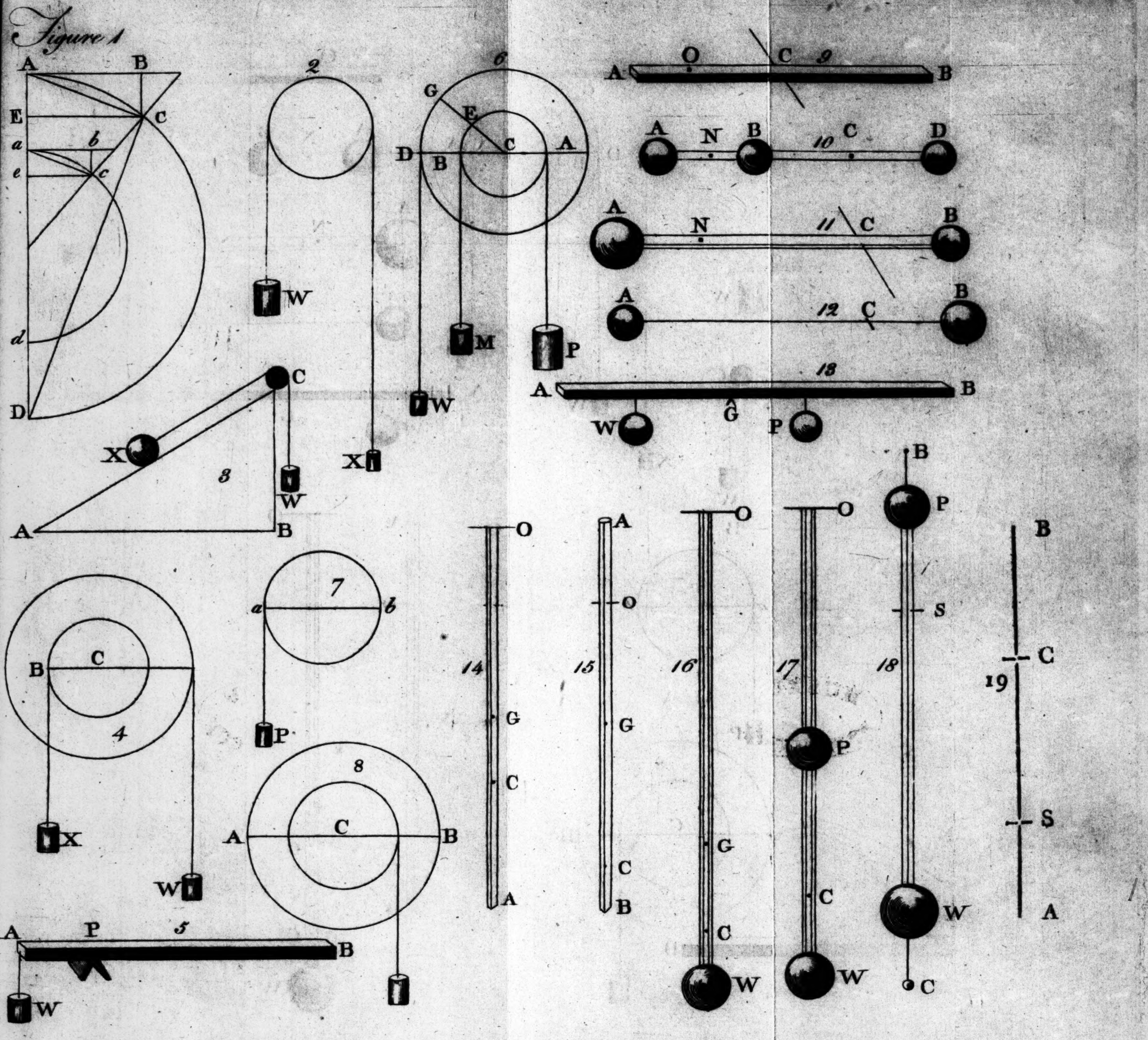
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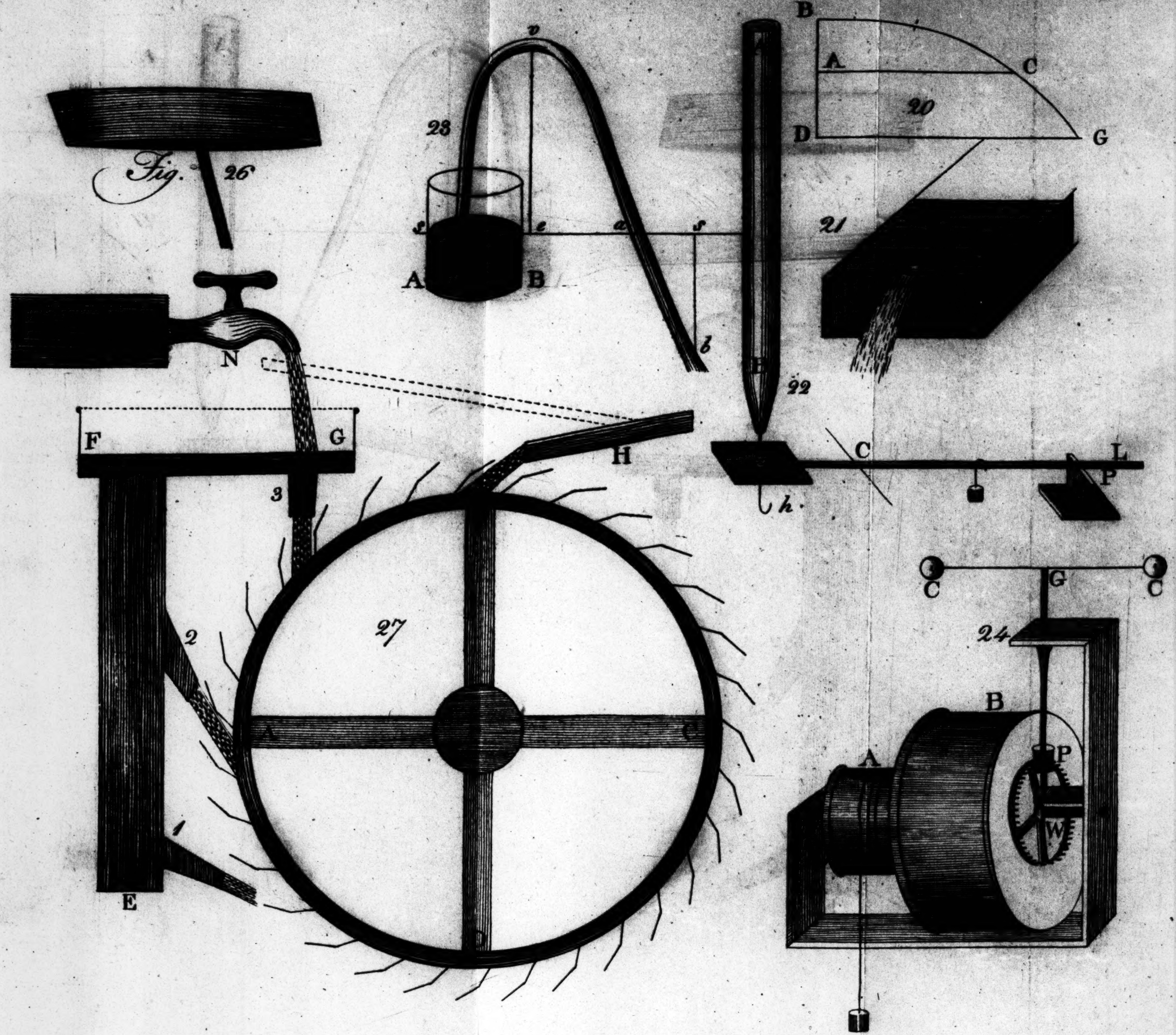
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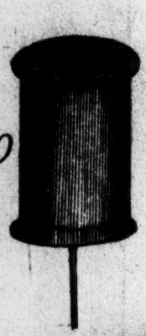

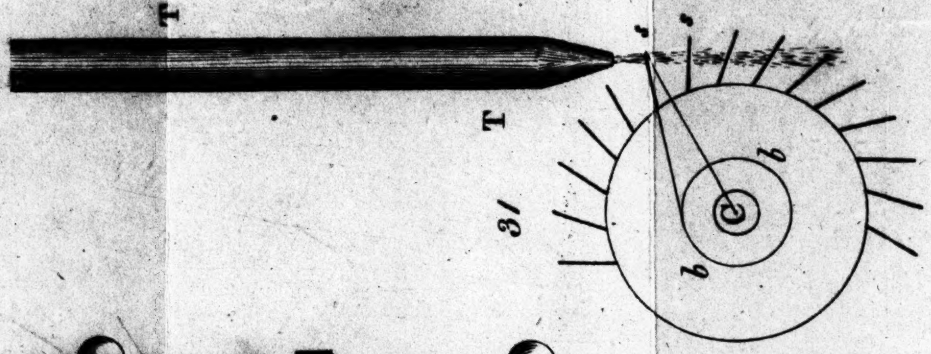

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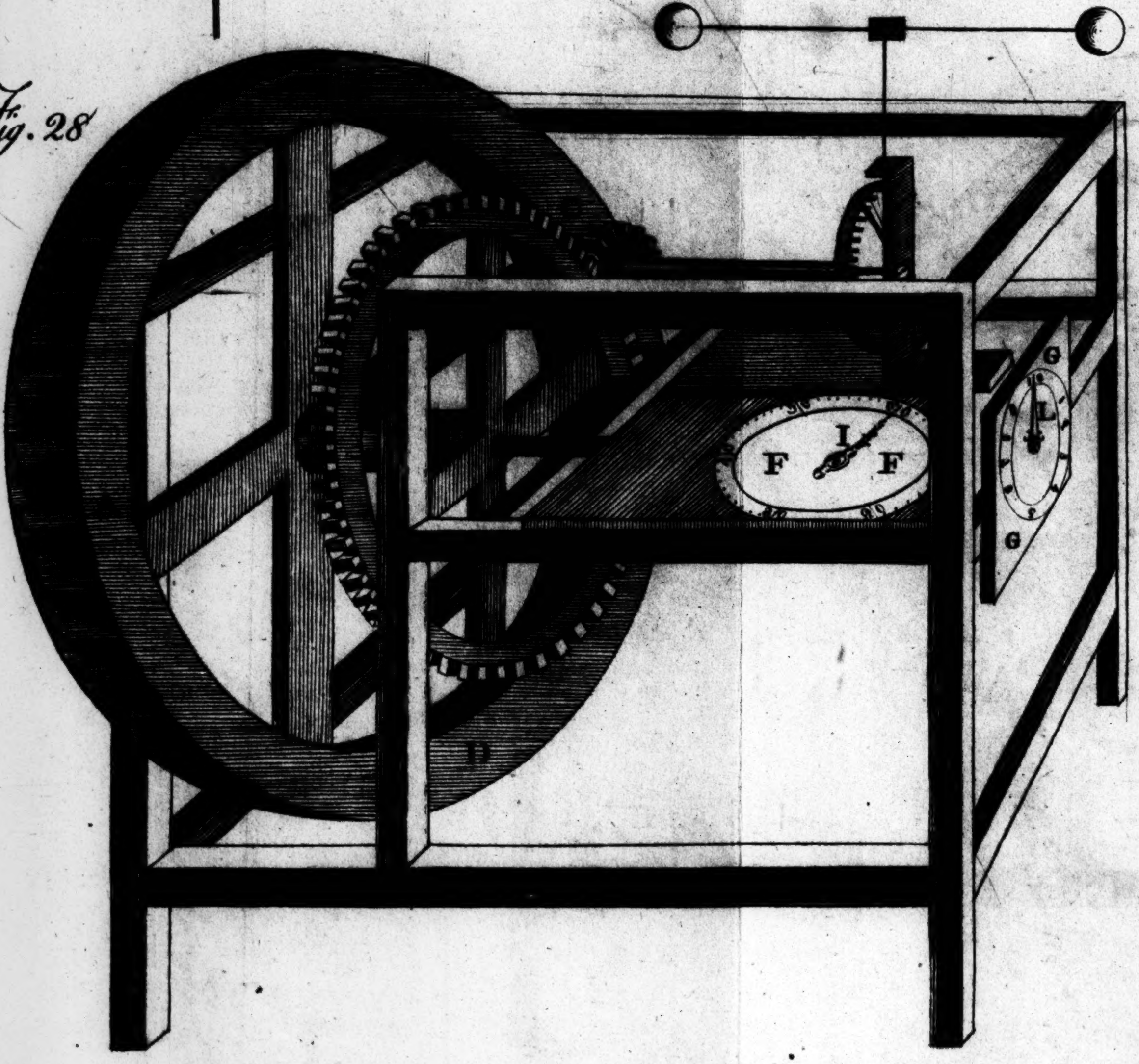
Fig. 28



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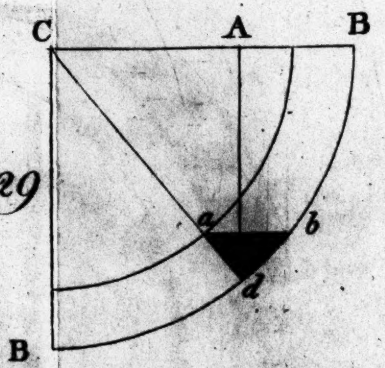

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